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# On the quantum density of states and partitioning an integer

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**Abstract**

This paper exploits the connection between the quantum many-particle density of states and the partitioning of an integer in number theory. For $N$ bosons in a one-dimensional harmonic oscillator potential, it is well known that the asymptotic ($N \to \infty$) density of states is identical to the Hardy–Ramanujan formula for the partitions $p(n)$, of a number $n$ into a sum of integers. We show that the same statistical mechanics technique for the density of states of bosons in a power-law spectrum yields the partitioning formula for $p^s(n)$, the latter being the number of partitions of $n$ into a sum of $s$th powers of a set of integers. By making an appropriate modification of the statistical technique, we are also able to obtain $d^s(n)$ for *distinct* partitions. We find that the distinct square partitions $d^2(n)$ show pronounced oscillations as a function of $n$ about the smooth curve derived by us. The origin of these oscillations from the quantum point of view is discussed. After deriving the Erdos–Lehner formula for restricted partitions for the $s = 1$ case, we use the modified technique to obtain a new formula for distinct restricted partitions.


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## 1. Introduction

The $N$-particle density of states of a self-bound or trapped system has attracted the attention of physicists for a long time. We have in mind the work done in nuclear [1] and particle physics [2], as well as in connection to black-hole entropy [3] in recent times. In nuclear physics, one is generally interested in self-bound fermions at an excitation energy $E$ that is large compared to the average single-particle level spacing, but is small compared to the fermi energy of the nucleus. In this energy range, the density of states is given by the highly successful Bethe formula [1] that grows as $\exp(a\sqrt{E})$, and is insensitive to the details of the single-particle spectrum. The constant $a$ in the exponent is proportional to the single-particle density of states at the fermi energy $E_{\mathrm{F}}$. In hadronic physics [2], the many-particle density of states grows exponentially with $E$, and leads to the concept of a limiting temperature. The same behavior is found to hold for a bosonic system like gluons in a bag [4].

It is well known that for ideal bosons in a one-dimensional harmonic trap, the asymptotic ($N \to \infty$) density of states is the same as the number of ways of partitioning an integer $n$ into a sum of other integers, and is given by the famous Hardy–Ramanujan formula [5]. It also grows exponentially as $\sqrt{E}$, the same as the Bethe formula when $E$ is identified with $n$. Grossmann and Holthaus [6] have studied this system, and have used more advanced results from the theory of partitions [7] to calculate the microcanonical number fluctuation from the ground state of the system as a function of temperature. Combinatorial methods have also been used to compute the thermodynamic functions for similar systems [8]. In this paper we use the $N$-particle quantum density of states (that may be derived using the methods of statistical mechanics) to obtain some novel results on the partitioning of an integer into a sum of squares, or a sum of cubes, etc. Some of the results pertaining to the partitions of an integer to a sum of *distinct* powers are new, and will be pointed out as they appear in the text. For the harmonic spectrum, we are also able to obtain the leading order finite $N$ (Erdos–Lehner [7]) correction to the asymptotic Hardy–Ramanujan formula using our method, and then get the corresponding (new) result for distinct restricted partitioning.

In Section 2 of this paper, we consider ideal bosons with a single-particle spectrum given by a sequence of numbers generated by $m^s$ ($m = 1, 2, 3, \ldots$) for a given integer $s \geqslant 1$. Such a spectrum has been studied before in a different context to examine the nearest neighbor spacings of a quantum many-body system [9]. To set the methodology, we first derive the asymptotic many-particle density of states for this system using the canonical ensemble and in the saddle-point approximation, and show that it grows exponentially as the $(s+1)$th-root of the excitation energy. Specifically, in the physically relevant case of a square well, this result implies that the density of states grows exponentially as the cube root of energy. Our general expression for the asymptotic density of states agrees with the Hardy–Ramanujan formula [5] for $p^s(n)$, the number of ways an integer $n$ may be expressed as a sum of $s$th powers of integers. Throughout this paper, we drop the superscript $s$ when $s = 1$.

We next extend our method to obtain asymptotically the number of distinct partitions $d^s(n)$ of an integer $n$ using the partition function of the fermionic particle spectrum (excluding the *hole* distribution, to be explained later). This analysis is presented in Section 3, where the smooth part of the asymptotic density of states (which reproduces the average behavior of the distinct partitions) is derived using the saddle-point approximation. While the asymptotic formula for distinct partition for $s = 1$ is known, we believe that our general formula for any $s$ given by Eq. (23) is new. Interestingly, for the $s = 2$ case where the integer $n$ is expressed as a sum of distinct squares of integers, computations of the exact values of $d^2(n)$ reveal large fluctuation about the smooth average curve. These fluctuations wax and wane in a beating pattern. The ratio of the amplitude of the oscillations to the smooth part of $d^2(n)$ goes to zero as $n \to \infty$. From the quantum angle, these oscillations in the many-particle density of states have their origin in the fluctuation of the degeneracy of the many-particle density of states about the average value. This, in turn, is related to the oscillatory part of the single-particle density of states of a one-dimensional square-well potential, and the constraint brought about by the Pauli principle.

In Section 4, we discuss the corrections to the saddle-point approximation when the number of particles $N$ is finite. In the theory of partitions this is known as restricted partitions (because of the upper limit on the number of partitions), as opposed to the unrestricted partitions discussed in Sections 2 and 3. We restrict our derivations to the harmonic oscillator ($s = 1$) spectrum in this section, since the canonical partition function is exactly known even for finite $N$ in this case. For the bosonic case, using this partition function, we derive exponentially small corrections for finite $N$, a result that agrees with the Erdos–Lehner [7] asymptotic formula for $s = 1$. This method is then extended for finding the finite $N$ correction for *distinct* partitions, a result that to our knowledge is new. We conclude with a summary of the main results.

## 2. The many-particle density of states

We first discuss the general statistical mechanical formulation for a $N$-particle system. The canonical $N$-particle partition function is given by

$$Z_N(\beta) = \sum_{E_i^{(N)}} \eta_i \exp(-\beta E_i^{(N)}) = \int_0^\infty \rho_N(E) \exp(-\beta E)\,\mathrm{d}E, \tag{1}$$

where $\beta$ is the inverse temperature, $E_i^{(N)}$ are the eigenenergies of the $N$-particle system with degeneracies $\eta_i$, and $\rho_N(E) = \sum_i \eta_i \delta(E - E_i^{(N)})$ is the $N$-particle density of states. The density of states $\rho_N(E)$ may therefore be expressed through the inverse Laplace transform of the canonical partition function

$$\rho_N(E) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \exp(\beta E) Z_N(\beta)\,\mathrm{d}\beta. \tag{2}$$

In general, it is not always possible to do this inversion analytically. Note that the single-particle density of states may be decomposed into an average (smooth) part, and oscillating components [10]. This, in turn, results in a smooth part $\bar{\rho}_N(E)$, and an oscillating part $\delta\rho_N(E)$ [11] for the $N$-particle density of states:

$$\rho_N(E) = \bar{\rho}_N(E) + \delta\rho_N(E). \tag{3}$$

The smooth part $\bar{\rho}_N(E)$ may be obtained by evaluating Eq. (2) using the saddle-point method [12]. Unlike the one-particle case, where the oscillating part may be obtained using the periodic orbits in a "trace formula" [10], it remains a challenging task to find an expression for the oscillating part $\delta\rho_N(E)$ [11]. In what follows, we shall use the saddle-point method to obtain the smooth asymptotic $\bar{\rho}_N(E)$, and identify it with the Hardy–Ramanujan formula for $p^s(n)$.

Before doing this, we note that the canonical partition function $Z_N(\beta)$ for a set of non-interacting particles with single-particle energies $\epsilon_i$, occupancies $\{n_i\}$, may also be written as

$$Z_N(\beta) = \exp\left(-\beta E_0^{(N)}\right) \sum_{\{n_i\}} \Omega(N, E_x) \exp(-\beta E_x\{n_i\}). \tag{4}$$

In the above, $E_0^{(N)}$ is the ground-state energy which we set to zero, and $E_x$ is the excitation energy. The sum is over the allowed occupation numbers for particles such that $E_x = \sum_i n_i \epsilon_i$. Note that for a given $E_x$, the number of excited particles in the allowed configurations may vary from one to a maximum of $N$. We denote by $\Omega(N, E_x)$ the total number of such distinct configurations allowed at an excitation energy $E_x$. We set the lowest single-particle energy at zero in order that $E_0^{(N)} = 0$, and consider a single-particle spectrum $\epsilon_m = m^s$. If now the excitation energy $E_x$ takes only integral values $n$, then $\Omega(N, E)$ is the same as the number of restricted partitions of $n$, denoted by $p_N^s(n)$, and asymptotically equivalent to the density of states $\bar{\rho}_N(E)$. Omitting the subscript $N$, as in $p^s(n)$, will imply that we are taking $N \to \infty$, corresponding to unrestricted partitioning.

To perform the saddle-point integration of Eq. (2), note that the integrand may be written as $\exp[S(\beta)]$, where $S(\beta)$ is the entropy given by,

$$S(\beta) = \beta E + \log Z_N. \tag{5}$$

Expanding the entropy around the stationary point $\beta_0$ and retaining only up to the quadratic term in the expansion in Eq. (2) yields the standard result [12]

$$\bar{\rho}_N(E) = \frac{\exp[S(\beta_0)]}{\sqrt{2\pi S''(\beta_0)}}, \tag{6}$$

where the prime denotes differentiation with respect to inverse temperature and

$$E = -\left(\frac{\partial \ln Z_N}{\partial \beta}\right)_{\beta_0}. \tag{7}$$

We now proceed with a single-particle spectrum given by $\epsilon_m = m^s$, where the integer $m \geqslant 1$, and $s > 0$ for a system of bosons. The energy is measured in dimensionless units. For example, when $s = 1$ the spectrum can be mapped on to the spectrum of a one-dimensional oscillator where the energy is measured in units of $\hbar\omega$. For $s = 2$, it is equivalent to setting energy unit as $\hbar^2/2m$, where is $m$ is the particle mass in a one-dimensional square well with unit length. These are the only two physically interesting cases. We, however, keep $s$ arbitrary even though for $s > 2$ there are no quadratic Hamiltonian systems. In particular $s$ need not even be an integer except to allow a comparison between the number theoretic results for $p_N^s(n)$ and the density of states $\rho_N(E)$ that we obtain here. We first obtain the asymptotic results for unrestricted partitioning by letting $N \to \infty$ and discuss the $N$-dependent correction later. The canonical partition function in this limit may be written as

$$Z_\infty(\beta) = \prod_{m=1}^{\infty} \frac{1}{[1 - \exp(-\beta m^s)]}, \tag{8}$$

where we have used the power-law form for the single-particle spectrum. By setting $x = \exp(-\beta)$, we see that the bosonic canonical partition function is nothing but the generating function [13] for $p^s(n)$ in number theory, the number of partitions of $n$ into perfect $s$th powers of a set of integers [5]:

$$Z_\infty(x) = \sum_{n=1}^{\infty} p^s(n) x^n = \prod_{n=1}^{\infty} \frac{1}{[1 - x^{n^s}]}. \tag{9}$$

In the limit $N \to \infty$, $p^s(n)$ is the same as $\Omega(E)$ where the energy $E$ is replaced by the integer $n$. In general the above form holds for all $s$ in the limit of $N \to \infty$, but is exact for finite $N$ only for the oscillator ($s = 1$) system [14,15]. Using Eqs. (5) and (8), and the Euler–MacLaurin series, we obtain

$$\begin{aligned} S &= \beta E - \sum_{n=1}^{\infty} \ln[1 - \exp(-\beta n^s)] \\ &= \beta E + \frac{C(s)}{\beta^{1/s}} + \frac{1}{2} \ln \beta - \frac{s}{2} \ln(2\pi) + O(\beta), \end{aligned} \tag{10}$$

where

$$C(s) = \Gamma\left(1 + \frac{1}{s}\right) \zeta\left(1 + \frac{1}{s}\right). \tag{11}$$

In the leading order, for determining the stationary point, we ignore the $\ln \beta$ term in the derivatives of $S$ and keeping only the dominant term we obtain

$$S'(\beta) = E - \frac{1}{s} \frac{C(s)}{\beta^{(1+1/s)}}. \tag{12}$$

Therefore the saddle-point is given by

$$\beta_0 = \left( \frac{C(s)}{sE} \right)^{s/(1+s)}. \tag{13}$$

The notation may be simplified by setting

$$\kappa_s = \left( \frac{C(s)}{s} \right)^{s/(1+s)}, \tag{14}$$

so that $\beta_0 = \kappa_s E^{-s/(1+s)}$. Substituting this value in the saddle-point expression for the density of states in Eq. (6)

$$\bar{\rho}^s_\infty(E) = \frac{\kappa_s}{(2\pi)^{(s+1)/2}}\sqrt{\frac{s}{s+1}}E^{-(3s+1)/(2(s+1))}\exp\left[\kappa_s(s+1)E^{1/(1+s)}\right]. \tag{15}$$

The RHS of the above equation is *identical* to that given for $p^s(n)$ in [5], the number of ways of expressing $n$ as a sum of integers with $s$th powers, if we replace $E$ by the integer $n$. For $s = 1$, for example, we have

$$\bar{\rho}_\infty(E) = \frac{\exp\left[\pi\sqrt{2E/3}\right]}{4\sqrt{3}E}, \tag{16}$$

which is simply the number of partitions of an integer $E$ in terms of other integers. For example $5 = 5$, $1 + 4$, $2 + 3$, $1 + 1 + 3$, $1 + 2 + 2$, $1 + 1 + 1 + 2$, and $1 + 1 + 1 + 1 + 1$, so $p(5) = 7$. Of course, the above asymptotic formula is not expected to be accurate for such a small integer, but it improves in accuracy for large numbers.

While the "physicists derivation" of the number partitions has been known for a while and indeed has been extensively used in the analysis of number fluctuation in harmonically trapped Bose gases [6], the derivation for a general power-law spectrum given above is novel even though the result was derived long ago by Hardy and Ramanujan [5] using more advanced methods. Equally interesting from the point of view of physics is the sensitivity of the bosonic density of states on the single-particle spectrum, in contrast to the fermionic Bethe formula. For example, where as in a harmonic well, both fermions and bosons have the exponential square-root dependence in energy for the density of states, as given in Eq. (16), in a square well only the fermions obey such a relation when the low temperature expansion is used. For the bosonic case, from Eq. (15), the density of states is

$$\bar{\rho}^2_\infty(E) = \sqrt{\frac{2}{3}}\frac{\kappa_2}{(2\pi)^{3/2}}\frac{\exp[3\kappa_2 E^{1/3}]}{E^{7/6}}. \tag{17}$$

This is the same as the asymptotic formula derived by Hardy and Ramanujan for the partition of $E$ into squares, for example $5 = 1^2 + 2^2, 1^2 + 1^2 + 1^2 + 1^2 + 1^2$. It is to be noted that in making the identification of $p^2(n)$ with $\rho^2_\infty(E)$, $E = n$ is to be identified as the excitation energy of the quantum system with a fictitious ground state at zero energy added to the square well.

In Fig. 1 we show a comparison between the exact (computed) $p(n)$ (continuous line), and $\bar{\rho}_\infty(n)$ (dashed line), as given by Eq. (16). We note that the Hardy–Ramanujan formula works well even for small $n$. Similarly, in Fig. 2, the computed $p^2(n)$ is compared with $\bar{\rho}^2_\infty(n)$, as given by Eq. (17). It will be noted from Fig. 2 that the computed $p^2(n)$ has step-like discontinuities, unlike the smooth behavior of $\bar{\rho}^2_\infty(n)$, specially for small $n$. We should remind the reader that these results are not new, and the corrections to the leading order Hardy–Ramanujan formula are also known in the number theory literature. We shall, however, obtain some new results using our method for distinct partitions $d^2(n)$ in the next section.

Before we conclude this section, we note that keeping terms of order $\beta$ in the saddle-point expansion of $S$ merely shifts the energy $E$ by the coefficient of the term proportional to $\beta$. For the $s = 1$ case in Eq. (10), there is indeed a term like $-\frac{1}{24}\beta$,

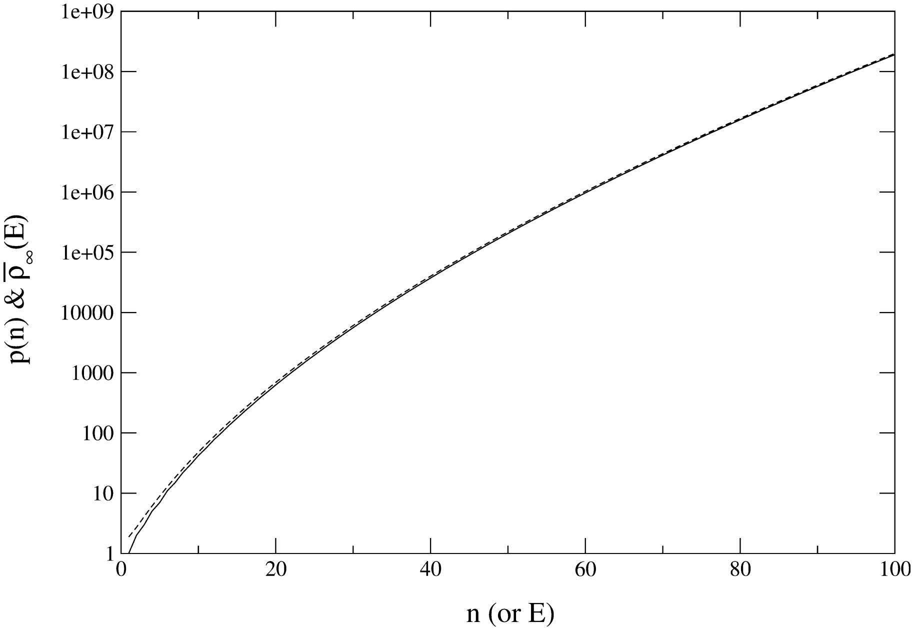


Fig. 1. Comparison of the exact $p(n)$ (solid line) and the asymptotic $\bar{\rho}_\infty(E)$ (dashed line), obtained from Eq. (16) for $s = 1$.

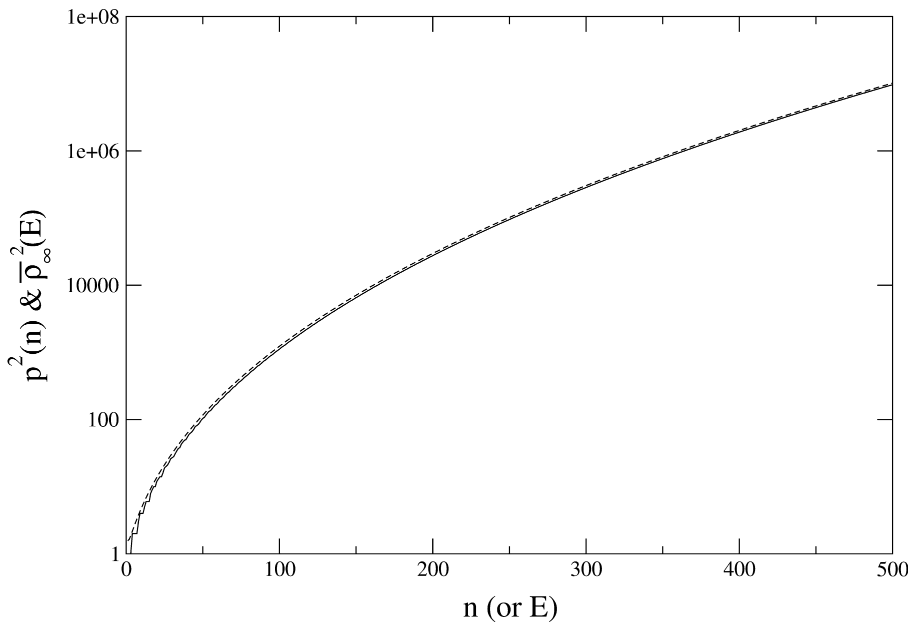


Fig. 2. Comparison of the exact $p^2(n)$ (solid line) and the asymptotic $\bar{\rho}^2_\infty(E)$ (dashed line), obtained from Eq. (17) for $s = 2$.

leading to the replacement of $E$ by $(E - \frac{1}{24})$ in Eq. (16). The resulting asymptotic expression for the density is the first term of the exact convergent series for partitions obtained by Rademacher [16]. Interestingly, for $s = 2$, there is no term of order $\beta$ in the Euler–MacLaurin expansion. A similar situation prevails for distinct partitions as will be shown in the next section.

## 3. Asymptotic density of states with distinct partitions

We now modify the method to obtain distinct partitions of an integer $n$ into $s$th powers, to be denoted by $d^s(n)$. For example, for $s = 1$, $n = 5$, the number of distinct integer partitions are 5, 2 + 3, and 1 + 4, so $d(5) = 3$. For distinct partitions, the first guess would be to use the fermionic partition function instead of the bosonic one of the previous section since distinctiveness of the parts is immediately ensured by the Pauli principle. However, there is a problem here which we illustrate using the $s = 1$ spectrum. For this case the fermionic partition function of non-interacting particles is given by (setting $x = \exp(-\beta)$ as before),

$$Z_N(x) = x^{N^{2/2}} \prod_{m=1}^{N} \frac{1}{(1 - x^m)} = x^{N^{2/2}} \sum_{n=0}^{\infty} \Omega(N, n) x^n, \tag{18}$$

which is the same as the bosonic partition function in a harmonic potential, except for the prefactor which is related to the ground-state energy of $N$ particles in the trap. Obviously, the $\Omega(N, n)$ is the same for both fermions and bosons even though $d_N(n)$ is different from $p_N(n)$. This is because the quantum mechanical ground state of fermions consists of occupied levels up to the fermi energy, unlike the bosons which all occupy the lowest energy state. Thus, for the fermions at any excitation energy, one should consider the distribution among particles as well as holes, each of which is separately distinct [17], and obey the Pauli principle. As we show below, the particle distribution at a given excitation energy measured from the Fermi energy identically reproduces (the unrestricted) but distinct partitions of an integer $n$, when $n$ is identified with the excitation energy.

The relevant "partition" function for the $m^s$ spectrum is given by,

$$\ln Z_\infty(\beta) = \sum_{m=1}^{\infty} \ln[1 + \exp(-\beta m^s)], \tag{19}$$

and the entropy $S(\beta)$ is obtained as usual by adding $\beta E$ to the above expression. Notice that this resembles the entropy of an $N$-fermion system, but with the chemical potential $\mu = 0$. In the normal $N$-fermion system at any given excitation energy the number of macro states available depends on the distribution of both particles above the Fermi energy and holes below the Fermi energy in the ground states. By setting $\mu = 0$ we are ignoring the hole distribution but only taking into account the states associated with the particle distribution. Because of Pauli principle implied in the above form for the entropy, only distinct partition of energy $E$ is allowed. Again, using the variable $x = \exp(-\beta)$ in Eq. (19), $Z_\infty(x)$ above is seen to be the generating function for distinct partitions $d^s(n)$ of an integer $n$ into $s$th powers of other integers [13].

Once this point is noted, the rest of the calculation proceeds as in the case of bosons and we obtain the following expression using the Euler–MacLaurin series

$$S(\beta) = \beta E + \frac{D(s)}{\beta^{1/s}} - \frac{1}{2}\ln(2) + \mathrm{O}(\beta), \tag{20}$$

where

$$D(s) = \Gamma\left(1 + \frac{1}{s}\right)\eta\left(1 + \frac{1}{s}\right), \tag{21}$$

where $\eta(s) = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l^s}$ denotes the alternating zeta function. Note that there is no $\log(\beta)$ term in Eq. (20). The saddle point $\beta_0$ is obtained by setting $S'(\beta_0) = 0$ as before. Defining

$$\lambda_s = (D(s)/s)^{s/(s+1)} \tag{22}$$

and using Eq. (6), we obtain

$$\bar{\rho}^s_{\infty(F)}(E) = \sqrt{s\lambda_s}\,\frac{\exp\left[(1+s)\lambda_s E^{1/(1+s)}\right]}{2\sqrt{\pi(1+s)E^{(2s+1)/(s+1)}}}, \tag{23}$$

where the subscript $(F)$ in $\bar{\rho}$ is to remind the reader that Fermi statistics has been used (with $\mu = 0$). Once again for $s = 1$ we recover the well-known asymptotic formula for the unrestricted but distinct partitions $d(n)$ of an integer [18], namely

$$\bar{\rho}_{\infty(F)}(E) = \frac{\exp[\pi\sqrt{E/3}]}{4 \times 3^{1/4}E^{3/4}}, \tag{24}$$

where, as usual, $E$ should be read as $n$. Similarly the asymptotic expression for $d^s(n)$ is given by in Eq. (23). We have not found this general expression in the literature. In Fig. 3, we show a comparison of the asymptotic density $\bar{\rho}_{\infty(F)}$ and the exact distinct partitions $d(n)$ of integer $n$ for $s = 1$. As in the case of bosonic partitions $p(n)$, the asymptotic formula for $d(n)$ works reasonably, except for $n < 10$. But the really interesting result is shown in Fig. 4 where we compare Eq. (23) for $s = 2$ with exact computations of $d^2(n)$. The asymptotic density of states follows the average of the exact $d^2(n)$ closely, but there are pronounced beat-like structure superposed on this smooth curve. This has come about because we have joined the computed points of $d^2(n)$ for discrete $n$'s by zig-zag lines. Note that compared to $d(n)$, the magnitude of $d^2(n)$ is very small, and this is one reason that the fluctuations in $d^2(n)$ look so prominent. We have checked numerically, however, that the ratio of the amplitude of the oscillations to its smooth average value decreases from about 1.5 to 0.2 as $n$ is increased to 1000. This means that for $n \to \infty$, the smooth part will eventually mask the fluctuations.

Although we cannot analytically reproduce these fluctuations in the many-particle density of states (or equivalently in $d^2(n)$), we can show from the quantum point of view that the smooth part $\bar{\rho}^2_{\infty(F)}$ arises strictly from the smooth part of the single-particle density of states. To make this point, let us derive the single-particle density of states, $g(\epsilon)$, for the $n^2$ spectrum. We begin with the knowledge of the exact single-particle spectrum, and write the canonical partition function:

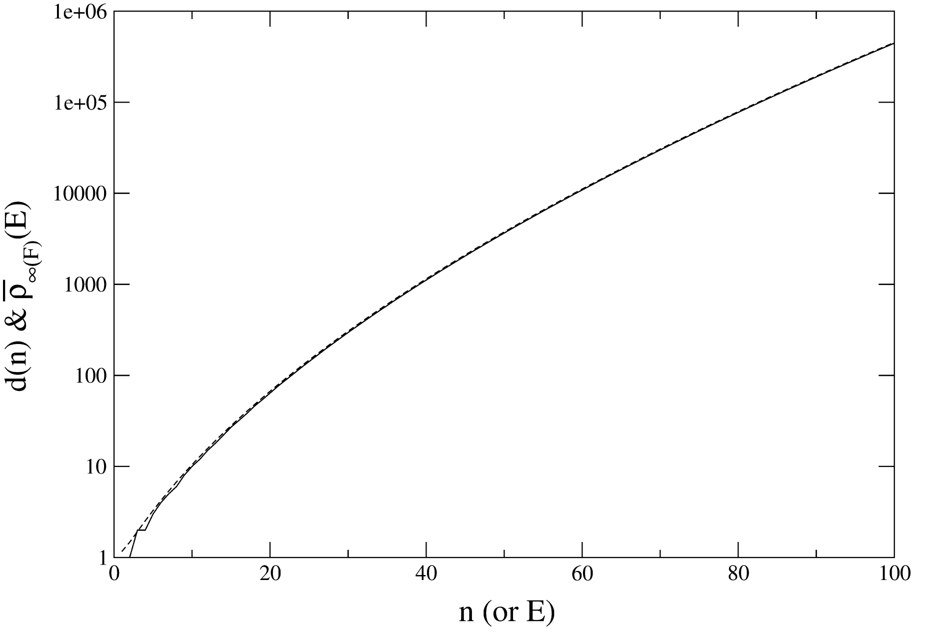


Fig. 3. Comparison of the exact $d(n)$ (solid line) and the asymptotic $\bar{\rho}_{\infty(F)}(E)$ (dashed line), obtained from Eq. (24) for $s=1$ and distinct partitions.

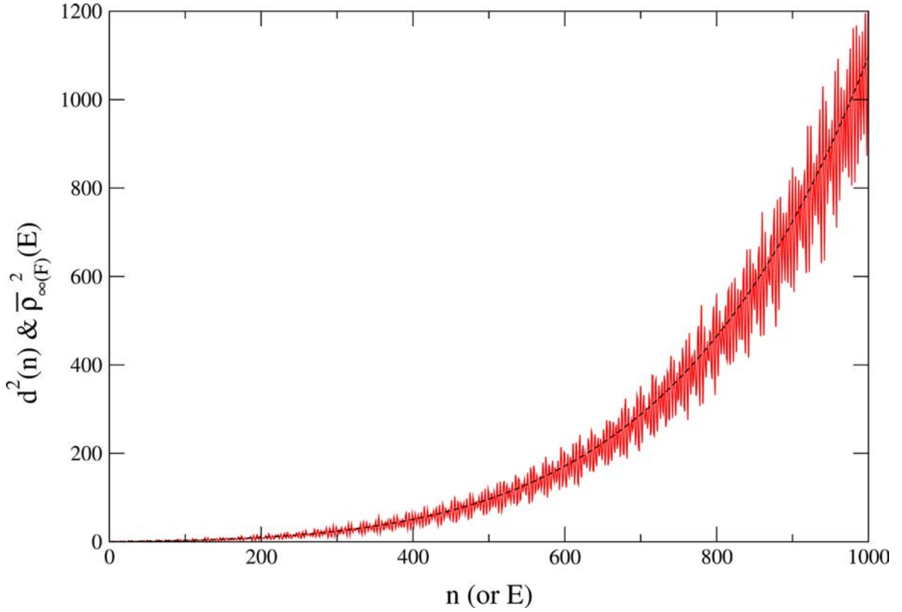


Fig. 4. Comparison of the exact $d^2(n)$ (solid line) and the asymptotic $\bar{\rho}^2_{\infty(F)}(E)$ (dashed line), obtained from Eq. (23) for $s=2$ and distinct partitions. Note that the $y$-axis is no longer in log scale.

$$Z_1(\beta) = \sum_{n=1}^{\infty} \exp(-\beta n^2). \tag{25}$$

To express this in a tractable form for Laplace-inverting, we use the (exact) Poisson sum formula

$$\sum_{n=-\infty}^{\infty} F(n) = \sum_{q=-\infty}^{\infty} \mathcal{F}(q), \tag{26}$$

where

$$\mathcal{F}(q) = \int_{-\infty}^{\infty} \mathrm{d}n\, F(n) \exp(2\pi \mathrm{i} q n). \tag{27}$$

Taking $F(n) = \exp(-\beta n^2)$ then gives $\mathcal{F}(q) = \sqrt{\pi/(\beta)} \exp(-\pi^2 q^2/(\beta))$. Using this result, we obtain

$$Z_1(\beta) = \frac{1}{2}\left(\frac{\pi}{\beta}\right)^{1/2} - \frac{1}{2} + \left(\frac{\pi}{\beta}\right)^{1/2} \sum_{q=1}^{\infty} \exp(-\pi^2 q^2/\beta). \tag{28}$$

On Laplace-inverting term by term, we obtain the exact result for the single-particle density of states:

$$g(\epsilon) = \frac{1}{2\sqrt{\epsilon}} - \frac{1}{2}\delta(\epsilon) + \frac{1}{\sqrt{\epsilon}} \sum_{q=1}^{\infty} \cos\left(2\pi \mathrm{q}\sqrt{\epsilon}\right), \tag{29}$$

$$= \overline{g}(\epsilon) + \delta g(\epsilon), \tag{30}$$

where $\overline{g}(\epsilon)$ is the "smooth" part consisting of the first two terms on the RHS of Eq. (29), and $\delta g(\epsilon)$ denotes the remaining oscillating terms. We can now evaluate Eq. (19) for $s = 2$ using the above $g(\epsilon)$:

$$\ln Z_\infty = \int_0^{\infty} g(\epsilon) \ln[1 + \exp(-\beta\epsilon)]\, \mathrm{d}\epsilon. \tag{31}$$

Evaluating the integrals, and adding $\beta E$ to it, we get the entropy

$$S(\beta) = \beta E + \frac{D(2)}{\beta^{1/2}} - \frac{1}{2}\ln(2) + \sqrt{\frac{\pi}{\beta}} \sum_{q=1}^{\infty} \sum_{l=1}^{\infty} \frac{(-)^{l+1}}{l^{3/2}} \exp\left(\frac{-\pi^2 q^2}{\beta l}\right). \tag{32}$$

We note that the first two terms on the RHS of the above equation are the same as obtained earlier in Eq. (20) using the Euler–MacLaurin expansion. These yielded the smooth many-body density of states given by Eq. (23) on using the saddle-point approximation. The term with the double sum in Eq. (32), which arise from $\delta g(\epsilon)$ in Eq. (30) and Fermi statistics, must be the source of the fluctuations seen in the density of states in Fig. 4 (the same $\delta g(\epsilon)$, when used in the bosonic case, gives a very different contribution to $S(\beta)$). In principle, exact Laplace inversion of $\exp[S(\beta)]$, where $S(\beta)$ is given by Eq. (32), should yield the fluctuating degeneracies of the quantum states with $E$, and hence of $d^2(n)$. We have not been able, however, to do this Laplace inversion. Since the oscillation in the exact partitions $d^2(n)$ resembles a beat-like structure, at least two frequencies must be interfering to give the pattern. Further work is needed to unravel this interesting point.

## 4. Finite size corrections, or restricted partitions

The smooth part of the many-particle density of states was derived in the previous sections for a system with $N \to \infty$, that corresponded to unrestricted partitions. We now apply the same method to obtain the asymptotic density of states for systems with finite size, that is when the number of particles is kept finite and equal to $N$. This corresponds to allowing the number of parts to be at most $N$. Consider, for example, for $s = 1, N = 4, n = 5$. Then, in restricted partitioning, the allowed partitions are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, and 2 + 1 + 1 + 1. The partition with 5 parts, 1 + 1 + 1 + 1 + 1 is not allowed, since the number of parts in this case is greater than 4. The above example is for restricted case that includes identical parts in a partition. These will be denoted by $p_N^s(n)$ in general, but for $s = 1$, the superscript will be dropped as usual. For the above example with restricted and distinct partitions, however, only 5, 1 + 4, and 2 + 3 are allowed. We denote such partitioning by $d_N^s(n)$ in general. In this section, we restrict to $s = 1$, and first present the leading order asymptotic expression for $p_N(n)$, using our method of calculating $\overline{\rho}_N(E)$. This result is already known in the literature by the Erdos–Lehner formula [7], but is derived here because we generalize it for obtaining the asymptotic expression for $d_N(n)$. To the best of our knowledge, this is a new result.

### 4.1. Asymptotic formula for $p_N(n)$

The $N$-boson canonical partition function in this case is exactly known:

$$\ln Z_N(\beta) = -\sum_{m=1}^{N} \ln[1 - \exp(-\beta m)]. \tag{33}$$

The canonical entropy $S_N$ is obtained as before by adding $\beta E$ to the above equation. Expanding the above using Euler–MacLaurin series, and assuming that $N$ is large so that $x = \exp(-\beta N) \ll 1$, even though $\beta \ll 1$. We then obtain

$$S_N(\beta) = S_\infty(\beta) - \exp(-\beta N)\left[\frac{1}{\beta} - \frac{1}{2}\right], \tag{34}$$

The stationary point is determined as before by the condition in Eq. (7) and for $N$ large it is the same as in Eq. (13). Substituting this in the saddle-point expression for the density of states in Eq. (6) we get

$$\overline{\rho}_N(E) = \overline{\rho}_\infty(E) \exp\left[-\left(\frac{\sqrt{6E}}{\pi} - \frac{1}{2}\right)\exp\left(-\frac{\pi N}{\sqrt{6E}}\right)\right]. \tag{35}$$

The above expression reproduces the well-known correction to the unrestricted partitions due to the restriction on the number of particles (see Erdos and Lehner [5]) apart from the constant term proportional to 1/2 in the exponent. This constant may, however, be neglected when $E$ is large. Using the conditions $\beta_0 \ll 1$ and $\beta_0 N \gg 1$, we see that formula (35) is valid in the region $C(1) \ll E \ll C(1)N^2$, where $C(1) \approx 1.645$. In Fig. 5 we compare the two differences, $[\overline{\rho}_\infty(E) - p_N(n)]$, and $[\overline{\rho}_N(E) - p_N(n)]$ for $N = 20$ (Fig. 5A), and $N = 30$ (Fig. 5B). In the above, $\overline{\rho}_\infty(E)$

is obtained from Eq. (16), $\overline{\rho}_{20}(E)$ is the Erdos and Lehner formula as given by Eq. (35), and $p_{20}(n)$ is the exact (computed) restricted partitions. Clearly, the former is much larger than the latter, indicating that Eq. (35) gives a better approximation to the exact values for restricted partitions.

### 4.2. Asymptotic formula for $d_N(n)$

Next we present the finding of an equivalent asymptotic formula to Eq. (35) for the restricted and distinct partition. This brings us back to the fermionic particle spectrum as discussed in Section 2 and [17]. Eq. (19) of Section 2 does not apply here, however, since it is applicable only for the unrestricted distinct partition, i.e.: $N \to \infty$. What we need is the exact canonical partition function for the particle space. From number theory [17,19], we found a formula for the (exact) number of ways of partitioning an integer $n$ to at most $N$ distinct parts:

$$d_N(n) = \sum_i^N p_i\left(n - \frac{i(i+1)}{2}\right), \tag{36}$$

where $p_i(n)$ is the (exact) number of partitions of $n$ to at most $i$ parts, which may be generated by the partition function given by Eq. (33). Eq. (36) implies that the partition or generating function for the restricted and distinct partition is given by:

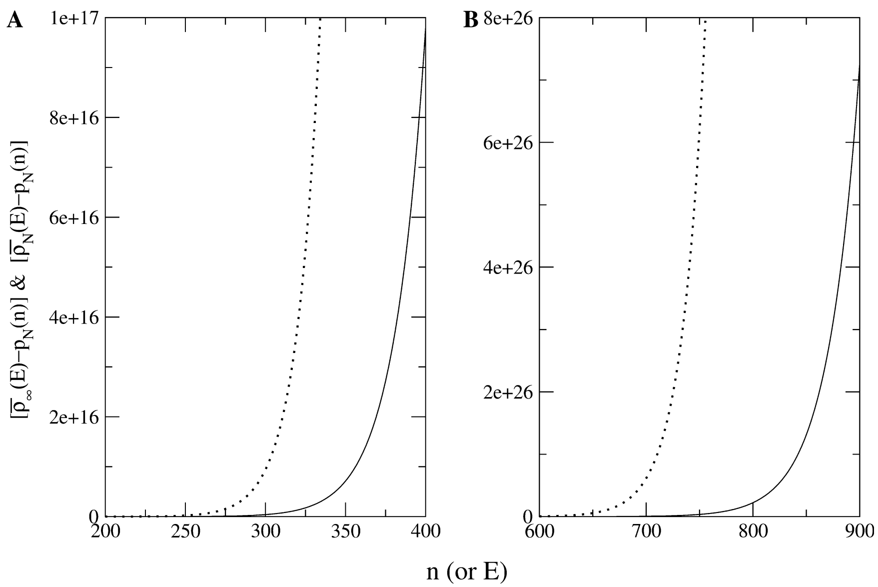


Fig. 5. (A) Comparison of $[\overline{\rho}_\infty(E) - p_{20}(n)]$ (dotted line) and $[\overline{\rho}_{20}(E) - p_{20}(n)]$ (solid line) for $N = 20$, where $\overline{\rho}_\infty(E)$ is obtained from Eq. (16), $\overline{\rho}_{20}(E)$ is the Erdos and Lehner formula as given by Eq. (35), and $p_{20}(n)$ is the exact (computed) restricted partitions. (B) Same for $N = 30$.

$$
\begin{aligned}
Z_N^{(d)}(\beta) &= \sum_{i=1}^{N} x^{i(i+1)/2} \prod_{n=1}^{i} \frac{1}{(1-x^n)} \\
&= \prod_{n=1}^{\infty} (1+x^n) - \sum_{i=N+1}^{\infty} x^{i(i+1)/2} \prod_{n=1}^{i} \frac{1}{(1-x^n)}. \qquad (37)
\end{aligned}
$$

The first term on the right-hand side of Eq. (37) is the generating function for the unrestricted distinct partition Eq. (19), and the second term is a sum of the generating functions for the restricted non-distinct partition Eq. (33) with the integer shifted to $i(i+1)/2$. To find an asymptotic formula for the restricted distinct partition $d_N(n)$, as usual, we take inverse Laplace transform of Eq. (37):

$$
\begin{aligned}
d_N(n) &= L_\beta^{-1}\left\{ \prod_{n=1}^{\infty}(1+x^n) \right\} - \sum_{i=N+1}^{\infty} L_\beta^{-1}\left\{ x^{\Delta} \prod_{n=1}^{i} \frac{1}{(1-x^n)} \right\} \\
&= d(n) - \sum_{i=N+1}^{\infty} p_i(n-\Delta) \sim \overline{\rho}_{\infty(F)}(E) - \sum_{i=N+1}^{\infty} \overline{\rho}_i(E-\Delta) \\
&= \frac{\exp[\pi\sqrt{E/3}]}{4\times 3^{1/4}E^{3/4}} - \sum_{i=N+1}^{\infty} \overline{\rho}_\infty(E-\Delta)\exp\left[ -\left( \frac{\sqrt{6(E-\Delta)}}{\pi} - \frac{1}{2} \right) \right. \\
&\quad \left. \times \exp\left( -\frac{\pi N}{\sqrt{6(E-\Delta)}} \right) \right] = \overline{\rho}_{N(F)}(E), \qquad (38)
\end{aligned}
$$

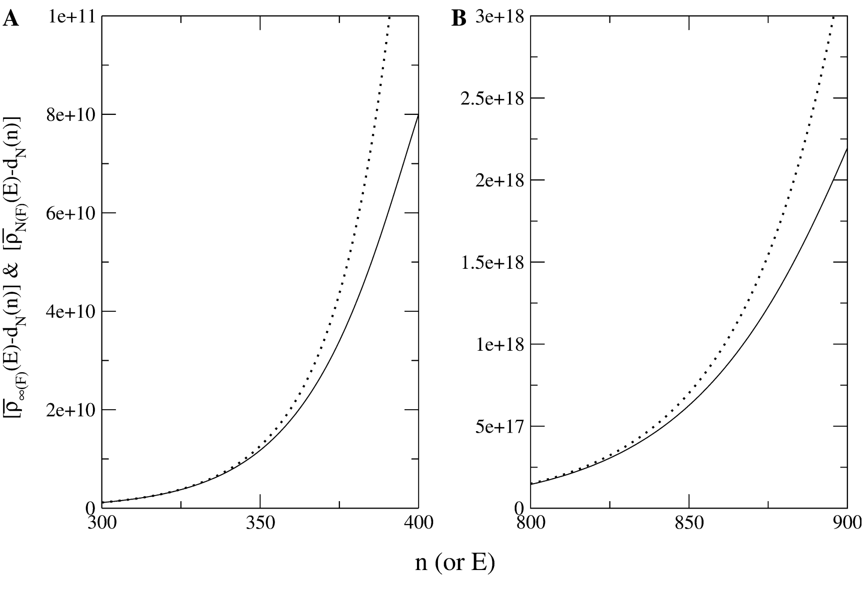


Fig. 6. (A) Comparison of $[\overline{\rho}_{\infty(F)}(E) - d_{20}(n)]$ (dotted line) and $[\overline{\rho}_{20(F)}(E) - d_{20}(n)]$ (solid line) for $N = 20$, where $\overline{\rho}_{\infty(F)}(E)$ is obtained from Eq. (24), $\overline{\rho}_{20(F)}(E)$ from Eq. (38), and $d_{20}(n)$ is the exact (computed) restricted distinct partitions. (B) Same for $N = 30$.

where $\Delta \equiv i(i+1)/2$, $x \equiv \exp(-\beta)$ and $n$ is identified with $E$. Note that since the asymptotic expression for the restricted partition $\bar{\rho}_N(E)$ is valid only for $C(1) \ll E \ll C(1)N^2$, Eq. (38) is thus valid only in this range. Fig. 6 displays the two differences, $[\overline{\rho}_{\infty(F)}(E) - d_N(n)]$, and $[\overline{\rho}_{N(F)}(E) - d_N(n)]$ for $N = 20$ (Fig. 6A), and $N = 30$ (Fig. 6B). In the above differences, $\overline{\rho}_{\infty(F)}(E)$ is obtained from Eq. (24), $\overline{\rho}_{N(F)}(E)$ from Eq. (38), and $d_N(n)$ is the exact (computed) restricted distinct partitions. Again, similar to the non-distinct case (Fig. 5), the $N$-correction asymptotic formula gives a better approximation to the exact finite $N$ partition than the infinite distinct one.

## 5. Discussion

This work emphasizes the connection between the many-body quantum density of states in a power-law spectrum with the number theoretic partitions $p^s(n)$, and the distinct partitions $d^s(n)$. This was already well known to the physics community for $p(n)$. While many of the results derived in this paper are known in the mathematical literature, the asymptotic formula for $d^s(n)$ (Eq. (23)), and the generalized formula (38) for restricted distinct partitions are, to the best of our knowledge, new. The fluctuations in $d^2(n)$, shown in Fig. 4, are interesting from the quantum mechanical point of view since these may be linked to the oscillating part of the density of states in a square-well potential, and the Pauli principle. However, we are not able to completely demonstrate this point to our satisfaction because of the difficulty of Laplace inversion of exponentiated quantities.

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