Intra-layer synchronization in multiplex networks

Lucia Valentina Gambuzza [1], Mattia Frasca [1], and Jesus Gomez-Gardeñes [2,3]
[1] DIEEI, Università degli Studi di Catania, Catania, Italy
[2] Departamento de Física de la Materia Condensada, University of Zaragoza, Zaragoza 50009, Spain
[3] Institute for Biocomputation and Physics of Complex Systems (BIFI), University of Zaragoza, Zaragoza 50018, Spain

We study synchronization of N oscillators indirectly coupled through a medium which is inhomogeneous and has its own dynamics. The system is formalized in terms of a multilayer network, where the top layer is made of disconnected oscillators and the bottom one, modeling the medium, consists of oscillators coupled according to a given topology. The different dynamics of the medium and the top layer is accounted by including a frequency mismatch between them. We show a novel regime of synchronization as intra-layer coherence does not necessarily require inter-layer coherence. This regime appears under mild conditions on the bottom layer: arbitrary topologies may be considered, provided that they support synchronization of the oscillators of the medium. The existence of a density-dependent threshold as in quorum-sensing phenomena is also demonstrated.

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Synchronization is one of the most ubiquitous collective phenomena appearing in natural and artificial systems [1, 2]. Singing crickets, fireflies emitting sequences of light flashes, cardiac pacemakers, circadian rhythms in mammals, firing neurons, chemical systems exhibiting oscillatory variation of the concentration of reagents, applauding audiences, or electrical and electronic devices are all common examples of systems operating in synchrony [3]. In general, all of these examples can be described as systems composed of many units that adjust a particular dynamical properties to behave in unison. The interaction among the units is at the core of synchronization since, when isolated, they behave according to their individual rhythms. In the recent years the way units interact and its influence to the onset of synchrony have been the subject of intense research, where complex networks have been used to account for a variety of interaction patterns [4, 5]. These patterns include the modeling of heterogeneity of links, delays in signal interchange, and time-dependent connections.

The main hypothesis underlying the network approach is that the units of a system (modeled as the nodes of a network) are directly coupled through interactions represented by the network edges [6, 7]. However, in many physical systems the units interact in an indirect way. For instance, in the Huygens's experiment, historically considered the first report on synchronization [8], the two pendulum clocks interact through the wooden beam on which they are both mounted. Similarly, communication between cellular populations occurs thanks to small molecules diffused in the medium [9], and chemical oscillators interact through a stirred solution [10, 11]. Even in the excessive wobbling observed in the opening of the Millennium Bridge in London, the synchronous pacing of the crowd derives from the interaction of the pedestrians with the bridge [12]. Synchrony in this case only occurs for a population density greater than a threshold, a phenomenon which is called as *crowd synchrony*.

Synchronization of indirectly coupled units has been studied in several works. The first evidences of synchronization through indirect coupling were observed in the context of quorum-sensing studies [11, 13, 14]. For instance, yeast cells, which show a density-dependent transition to synchronous oscillations, only interact by exchanging signaling molecules in the extracellular solution [14]. The studies about the synchronization of periodic oscillators coupled through a common medium have been recently extended to chaotic systems. In this latter case, when two chaotic units are considered, both in-phase and anti-phase synchronization have been numerically [15–17] and experimentally [18, 19] observed. When more than two chaotic units are taken into account, phenomena such as phase synchronization, periodic collective behavior and quorum-sensing transition show up [20, 21].

The general model for the study of N dynamical units coupled through a common medium is described by:

$$\begin{array}{l} \dot{x}_j = f(x_j) + \lambda_1 (z - x_j) \ , \\ \dot{z} = \frac{\lambda_2}{N} \sum_{j=1}^{N} (x_j - z) - Jz \end{array} \tag{1}$$

where x_j is the state vector of the $j-$th dynamical unit and z is that of the common medium. The dynamics of each unit is regulated by the function $f(x)$, that describes the internal dynamics of each unit when isolated, and the linear coupling with the medium, whose strength is λ_1. In its turn, the dynamics of the medium is activated by the coupling with the units (weighted by λ_2) and incorporates a decay term with coefficient J.

In Eq. (1) a homogeneous distribution of the medium is assumed. While this assumption is reasonable for chemical systems under the hypothesis of well-stirred solutions or biological systems under the hypothesis of fast diffusion of the small molecules, in other contexts (such as genetic oscillators [9]) the interactions may be mediated by one agent in the medium for each dynamical unit. Thus, a model in which units are not directly coupled, while the agents in the medium interact, is needed.

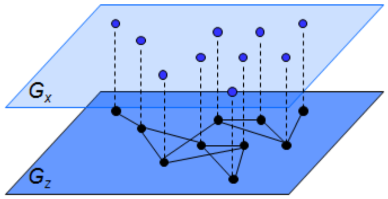


FIG. 1: Representation of the multiplex network consisting of two layers with one-to-one coupling between the layers. In the top layer, called x, the nodes only interact which those in the bottom one whereas in the bottom layer z the nodes also interact with other members of the same layer.

In this work, we consider this latter scenario and propose a dynamical model similar to Eq. (1) incorporating a microscopic description of the interactions of the agents in the medium. In particular, we account for the assumption of inhomogeneous and not passive environment by investigating a system made of two layers, one representing the medium, called layer z, and the other, called x, the dynamical units. The interaction between layers (medium and units) is as schematically shown in Fig. 1. Each unit interacts directly with one agent in the medium. Therefore, in terms of the recently developed theory of multilayer networks [22] , our system is termed as a *multiplex network* of two layers.

Multiplexes have recently attracted a lot of attention as they are the kind substrates representing better the interaction patterns occurring in many dynamical processes such as diffusion [23, 24], congestion and traffic [25], evolutionary dynamics [26] or epidemics [27]. To extend this knowledge to the realm of synchronization we assume that each node of the multiplex is a Stuart-Landau oscillator with different natural frequency. In this way the multiplex is formed by N Stuart-Landau oscillators coupled according to the adjacency matrix A^z_{ij} in the layer z, and N Stuart-Landau oscillators in the layer x that are not directly coupled. Thus, the evolution of the state, $u^\alpha_j \in \mathbb{C}$, of oscillator j in layer $\alpha(= x, z)$, is given by:

$$\begin{array}{rcl}\dot{u}^x_j & = & (a + \mathrm{i}\cdot\omega^x_j - |u^x_j|^2)\cdot u^z_j + \lambda_{zx}\cdot(u^z_j - u^x_j)\ , \\ \dot{u}^z_j & = & (a + \mathrm{i}\cdot\omega^z_j - |u^z_j|^2)\cdot u^z_j + \lambda_{zx}\cdot(u^x_j - u^z_j) \\ & & +\lambda_z\cdot\sum_{l=1}^{N} A^z_{jl}(u^z_l - u^z_j)\ ,\end{array} \tag{2}$$

where $\sqrt{a}$ and ω^α_j are respectively the amplitude and the frequency of oscillator j when uncoupled, and λ_z and λ_{zx} are the coupling between the agents in the medium and the inter-layer coupling respectively. The natural frequencies, ω^α_j, of the nodes are uniformly distributed in $[0.95\cdot\omega^\alpha, 1.05\cdot\omega^\alpha]$, being $\omega^x = 1$ and $\omega^z = 2.5$. Let us note that the Stuart-Landau model considered here contains the Kuramoto model [28] (the usual benchmark for the study of synchronization in networks) as a limiting case when the amplitude dynamics, which occurs when a is large. Networks of Stuart-Landau oscillators with a heterogeneous distribution of the natural frequencies have been also studied in networks [29, 30].

We now investigate the existence of phase synchronization in the multiplex. Our aim is to show that, besides global synchronization, *i.e.*, the regime in which all the network nodes are synchronized each other, a state characterized by intra-layer coherence and inter-layer incoherence is possible. We refer to this regime as intra-layer coherence (ILC), implicitly assuming that there is no coherence between the two layers (otherwise global synchronization is obtained). Note that the regime showing ILC is rather counterintuitive since all the oscillators in each layer oscillate in synchrony with a shared frequency Ω^α which is different from one layer to the other. Moreover, as the are no intra-layer connections in layer x, the synchronization of this layer is possible due to the indirect coupling through layer z. Therefore, in the ILC regime, the nodes in layer z are mediating for synchronization of layer x nodes, without being synchronized with them.

We first analytically show the existence of ILC regime by rewriting the system (2) in polar coordinates ($u^\alpha_j = \rho^\alpha_j \exp \mathrm{i}\theta^\alpha_j$) and focusing on the equations for the phases:

$$\begin{array}{l}\dot{\theta}^x_j = \omega^x_j + \lambda_{zx}\frac{\rho^z_j}{\rho^x_j}\sin(\theta^z_j - \theta^x_j)\ , \\ \dot{\theta}^z_j = \omega^z_j + \lambda_{zx}\frac{\rho^x_j}{\rho^z_j}\sin(\theta^x_j - \theta^z_j) + \lambda_z\sum_{l=1}^{N} A^z_{jl}\frac{\rho^z_l}{\rho^z_j}\sin(\theta^z_l - \theta^z_j)\ .\end{array} \tag{3}$$

We look for solutions of the type $\theta^z_1 = \theta^z_2 = \ldots = \theta^z_N$ and $\theta^x_1 = \theta^x_2 = \ldots = \theta^x_N$, *i.e.*, solutions where all the oscillators within each layer are synchronized (this condition includes both the regimes of ILC and global synchronization). Under this hypothesis, we consider two generic nodes j and l in Eqs. (3), and we also assume that the frequency of the two nodes are similar to derive:

$$\begin{array}{l}\frac{d}{dt}(\theta^x_j - \theta^x_l) = \lambda_{zx}(\frac{\rho^z_j}{\rho^x_j} - \frac{\rho^z_l}{\rho^x_l})\sin(\theta^z_j - \theta^x_j)\ , \\ \frac{d}{dt}(\theta^z_j - \theta^z_l) = \lambda_{zx}(\frac{\rho^x_j}{\rho^z_j} - \frac{\rho^x_l}{\rho^z_l})\sin(\theta^x_j - \theta^z_j)\ .\end{array} \tag{4}$$

From these equations we notice that a solution corresponding to global synchronization of the multiplex, $\theta^z_1 = \ldots = \theta^z_N = \theta^x_1 = \ldots = \theta^x_N$, is always possible. However, the solution corresponding to ILC, $\theta^z_1 = \ldots = \theta^z_N = \theta^z$ and $\theta^x_1 = \ldots = \theta^x_N = \theta^x$ with $\theta^z - \theta^x \neq$ const., is only possible provided $\frac{\rho^z_1}{\rho^x_1} = \frac{\rho^z_2}{\rho^x_2} = \ldots = \frac{\rho^z_N}{\rho^x_N}$, *i.e.*, the nodes in the same layer must have the same amplitude. Thus, by fixing $\rho^z_j = \rho^z\ \forall j$ and $\rho^x_j = \rho^x\ \forall j$, and by looking at the equations of the amplitudes, it is possible to show that the ILC solution cannot be achieved with a stationary amplitude, $\dot{\rho}^\alpha = 0$ ($\alpha = x, z$), *i.e.*, it cannot be observed in a multiplex composed of Kuramoto oscillators. Under this hypothesis, the equations for the amplitude are:

$$\begin{array}{l}\dot{\rho}^x = a\rho^x - (\rho^x)^3 + \lambda_{zx}\left[\rho^z\cdot\cos(\theta^z - \theta^x) - \rho^x\right]\ , \\ \dot{\rho}^z = a\rho^z - (\rho^z)^3 + \lambda_{zx}\left[\rho^x\cdot\cos(\theta^x - \theta^z) - \rho^z\right]\ .\end{array} \tag{5}$$

From above it becomes clear that a stationary solution ($\dot{\rho}^x = \dot{\rho}^z = 0$) of Eqs. (5) implies that $\rho^x = \rho^z$, *i.e.*, all

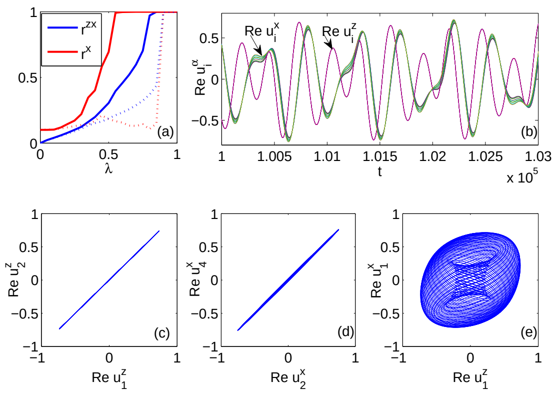


FIG. 2: (Color online). The ILC regime in a multilayer network with $N = 10$ nodes and all-to-all coupling in the bottom layer z. (a) Kuramoto order parameters r^x and r^{zx} vs. $\lambda = \lambda_{zx} = \frac{\lambda_z}{5}$. Continuous lines refer to a multilayer network of Stuart-Landau oscillators with $a = 1$, whereas dashed ones to purely phase oscillators ($a \to \infty$), for which ILC does not exist. (b) Waveforms of state variables $\mathrm{Re}\, u_j^\alpha$ for $\lambda = 0.7$. (c) Phase plane $\mathrm{Re}\, u_1^z - \mathrm{Re}\, u_2^z$. (d) Phase plane $\mathrm{Re}\, u_2^x - \mathrm{Re}\, u_4^x$. (e) Phase plane $\mathrm{Re}\, u_1^z - \mathrm{Re}\, u_1^x$. In (b)-(e) nodes in each layer are mutually synchronized with the nodes of the same layers, but not with their corresponding counterpart in the other layer.

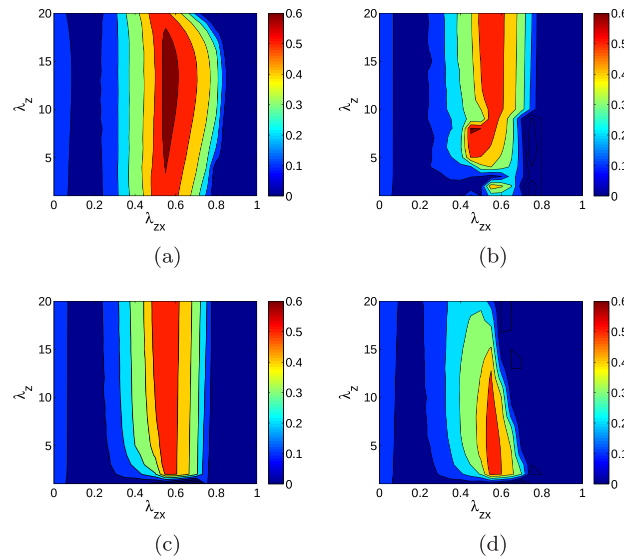


FIG. 3: (Color online). Bifurcation diagram of Δr vs. λ_z and λ_{zx} for multiplex networks with $N = 100$ and several interaction topologies for the bottom layer z: (a) all-to-all network; (b) lattice; (c) SF network; (d) ER network.

the nodes having the same amplitude, and

$$\cos(\theta^x - \theta^z) = \frac{\lambda_{zx} + (\rho^x)^2 - a}{\lambda_{zx}} , \tag{6}$$

i.e., the difference $\theta^z - \theta^x$ is constant, contrary to the initial hypothesis. This result points out that the solution cannot be stationary (as it requires time-varying amplitudes) and thus can be only obtained when the amplitude is a free parameter. This condition is met in Stuart-Landau oscillators, but not in Kuramoto ones.

We now provide numerical evidences of the existence of ILC solutions in a small network and, then, examine the case of larger systems. Phase synchronization between any pair of oscillators of the multiplex, namely oscillator j of layer α and oscillator l of layer β, can be measured by the Kuramoto order parameter

$$r_{jl}^{\alpha\beta} = |\langle e^{\mathrm{i}\cdot[\theta_j^\alpha(t) - \theta_l^\beta(t)]}\rangle_t| . \tag{7}$$

To get some insight on the behavior of the layers we monitor the intra-layer coherence by defining the Kuramoto order parameter of layer α as

$$r^\alpha = \frac{1}{N(N-1)} \sum_{j,l=1}^{N} r_{jl}^{\alpha\alpha} , \tag{8}$$

and the inter-layer coherence as

$$r^{zx} = \frac{1}{N} \sum_{j=1}^{N} r_{jj}^{zx} , \tag{9}$$

i.e., by averaging the degree of synchronization between all the pairs of nodes connected by the inter-layer links.

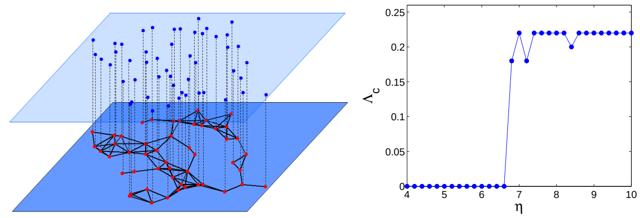
In Fig. 2 we show the results obtained with a multiplex of $N = 10$ units in each layer, where the nodes in layer z are globally connected ($A_{jl}^z = 1\ \forall i, l$). The network behavior depends on the coupling coefficients λ_z and λ_{zx}. In this first example the analysis was carried out by simultaneously varying them and keeping their ratio constant, *i.e.*, we varied λ defined as $\lambda = \lambda_{zx} = \frac{\lambda_z}{5}$ (the more general case of independent coupling coefficients is considered below). The Kuramoto order parameters r^x and r^{zx} vs. λ [see Fig. 2(a)] show that r^x grows faster than r^{zx}, and consequently there is a range of λ values for which the top layer reaches synchronization (r_x close to unit), even if each node of the top layer is not synchronized to its corresponding in the bottom layer (r_{zx} small). When purely phase (Kuramoto) oscillators are considered, this latter regime is not observed as the curves of r_x and r_{zx} coincide [dashed lines in Fig. 2.(a)]. The waveforms obtained for $\lambda = 0.7$ [see Fig. 2(b)] confirm that the ILC regime is only attainable together with non-stationary amplitudes. Fig. 2(b) also shows that the nodes in each layer are synchronized with a frequency different from one layer to the other. Intra-layer synchronization without inter-layer coherence is also clear from the phase planes of Figs. 2(c)-(e).

We now consider a larger multiplex networks with layers of $N = 100$ nodes and analyze different types of interaction topologies for layer z. We have investigated four kinds of undirected, unweighted networks: all-to-all coupling, regular lattices, scale-free (SF) and Erdös-Rényi (ER) networks. For these four networks we have monitored the difference between the Kuramoto order parameters in Eqs. (8) and (9), *i.e.*, $\Delta r = r^x - r^{zx}$, as a function of the two coupling coefficients λ_z and λ_{zx}. Large values of Δr indicate the appearance of the ILC regime in a region of the parameter space. With the exception of the all-to-all topology, the networks have the same average degree, $\langle k \rangle = 8$. As can be observed in Fig. 3 the ILC regime appears in all the cases. The only difference is that the region of the parameter space characterized by ILC differs from network to network, being the largest area in the case of all-to-all coupling. This finding points out the generality of the ILC regime in multiplex networks.

Finally, given the biological/chemical examples in which the model proposed applies, we have studied the influence that the density of agents in the media has on the onset of ILC. Our aim is to find the onset of a density-dependent threshold, in a similar fashion to those quorum sensing-like transitions to synchronization, that is typically induced by the indirect coupling provided by the medium. In particular, we have considered a multiplex network in which the topology of layer z is defined by a random geometric graph, *i.e.*, a spatial graph in which the nodes are randomly distributed in a planar space of size $L \times L$ with a density given by $\eta = \frac{N}{L^2}$ and each pair of nodes is connected only if their Euclidean distance is less or equal than a threshold r [see Fig. 4(a)].

To monitor the onset of a fully developed regime of ILC as function of the density of the particles in the medium, η, we have run simulations at a fixed value of λ_z while varying λ_{zx} and defined the following parameter. Starting from the typical scenario of ILC shown in Fig. 2(a), we observe that r^x reaches values close to one before r^{zx}, and that a measure of the existence of ILC is given by the large difference in the values of λ_{zx} for which r^x and r^{zx} approach one. We have thus defined λ^1_{zx} as $\lambda^1_{zx} = \min\{\lambda_{zx} : r_{zx}(\lambda_{zx}) > 0.95\}$ and λ^2_{zx} as $\lambda^2_{zx} = \min\{\lambda_{zx} : r_x(\lambda_{zx}) > 0.95\}$, and monitor the difference between these two values, indicated as $\Lambda_c = \lambda^2_{zx} - \lambda^1_{zx}$. Fig. 4(b) reports the trend of Λ_c as a function of the density η, clearly showing the existence of a density-dependent threshold, η_c, for the appearance of ILC. Below the $\eta_c \simeq 6.6$ no ILC is observed, while above this threshold ILC develops after a very sharp transition.

In summary, we have analyzed synchronization in a system of N oscillators indirectly coupled through a inhomogeneous medium. In addition, the medium has its own dynamics, which is of the same type of the oscillators (periodic, when uncoupled), but with a different natural oscillation frequency. The system has been modeled as a multiplex network formed by two layers with the same number of nodes, whereas the nodes of a layer are connected in a one-to-one correspondence with those of the other layer. We have shown the onset of intra-layer synchronization without inter-layer coherence. This regime is commonly observed independently from the topology of the layer corresponding to the medium, although the exact region in the parameter space in which it appears depends on it. We have shown that the presence of an amplitude dynamics is fundamental as the regime if intra-layer synchrony is not observed in purely phase oscillators. Finally, we have shown the onset of a density-dependent threshold, characteristic of crowd synchrony phenomenon, when the topology of the indirect coupling is inherited by a random geometric graph, thus recovering the spatial nature of a medium in chemical and biological systems.



(a) (b)

FIG. 4: (Color online). (a) An example of a multilayer network where the bottom layer is a random geometric graph. For the sake of visualization a network with only $N = 50$ nodes is displayed. (b) Behavior of the parameter Λ_c as a function of the density η for a multilayer network where the bottom topology is defined by a random geometric graph. The network has $N = 100$ nodes and the coupling coefficient in the bottom layer is fixed to $\lambda_z = 2$.

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[1] A. Pikovsky, M. Rosenblum, & J. Kurths, *Synchronization: a Universal Concept in Nonlinear Sciences*, (Cambridge University Press, Cambridge, 2003).
[2] S. Boccaletti, *The Synchronized Dynamics of Complex Systems*, (Elsevier, Amsterdam, 2008).
[3] S. Strogatz, *SYNC. The emerging science of spontaneous order*, (Hyperion, New York, 2003).
[4] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, & D. U. Hwang, *Phys. Rep.* **424**, 175 (2006).
[5] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, & C.-S. Zhou, *Phys. Rep.* **469**, 93 (2008).
[6] R. Albert, and A.-L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).
[7] M.E.J. Newman, *SIAM Rev.* **45**, 167 (2003).
[8] M. Bennett, M. F. Schatz, H. Rockwood, and K. Wiesenfeld, *Proc. Roy. Soc. London A* **458**, 563 (2002).
[9] D. McMillen, N. Kopell, J. Hasty, and J. J. Collins, *Proc. Natl. Acad. Sci. USA* **99**, 679 (2002).
[10] R. Toth, A. F. Taylor, and M. R. Tinsley, *J. Phys. Chem. B* **110**, 10170 (2006).

[11] A. F. Taylor, M. R. Tinsley, F. Wang, Z. Huang, and K. Showalter, *Science* **323**, 614 (2009).
[12] S. H. Strogatz, D. M. Abrams, A. McRobie, B. Eckhardt, and E. Ott, *Nature*, **438**(7064), 43 (2005).
[13] J. Garcia-Ojalvo, M. B. Elowitz, and S. H. Strogatz, *Proc. Natl. Acad. Sci. USA* **101**, 10955 (2004).
[14] S. De Monte, F. d'Ovidio, S. Dano, and P. G. Sorensen, *Proc. Natl. Acad. Sci. USA* **104**, 18377 (2007).
[15] V. Resmi, G. Ambika, and R. E. Amritkar, *Physical Review E* **81**, 046216 (2010).
[16] A. Sharma, M. D. Shrimali, and S. K. Dana, *Chaos* **22**, 023147 (2012).
[17] A. Sharma, P. R. Sharma, and M. D. Shrimali, *Phys. Lett. A* **376**, 1562 (2012).
[18] A. Sharma, and M. D. Shrimali, *Pramana* **77**, 881 (2011).
[19] R. Suresh, K. Srinivasan, D. V. Senthilkumar, K. Murali, M. Lakshmanan, and J. Kurths, *Int. J. Bif. Chaos* **24**, 1450067 (2014).
[20] G. Russo, and J. J. E. Slotine, *Phys. Rev. E* **82**, 041919 (2010).
[21] B. W. Li, C. Fu, H. Zhang, and X. Wang, *Phys. Rev. E* **86**, 046207 (2012).
[22] M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivela, Y. Moreno, M.A. Porter, S. Gomez, and A. Arenas, *Phys. Rev. X* **3**, 041022 (2013).
[23] S. Gómez, A. Diaz-Guilera, J. Gómez-Gardeñes, C.-J. Perez-Vicente, Y. Moreno, and A. Arenas, *Phys. Rev. Lett.* **110**, 028701 (2013).
[24] F. Radicchi, and A. Arenas, *Nature Phys.* **9**, 717 (2013).
[25] R. G. Morris, and M. Barthelemy, *Phys. Rev. Lett.* **109**, 128703 (2012).
[26] J. Gómez-Gardeñes, I. Reinares, A. Arenas, and L.-M. Floría, *Sci. Rep.* **2**, 620 (2012).
[27] C. Granell, S. Gómez, and A. Arenas, *Phys. Rev. Lett.* **111** 128701 (2013).
[28] J. A. Acebrón, L. L. Bonilla, C. J. Pérez-Vicente, F. Ritort, and R. Spigler, "The kuramoto model: a simple paradigm for synchronization phenomena," *Rev. Mod. Phys.*, **77**, pp. 137, (2005).
[29] A. Bergner, M. Frasca, G. Sciuto, A. Buscarino, E. J. Ngamga, L. Fortuna, and J. Kurths, *Phys. Rev. E* **85**, 026208 (2012).
[30] L.V. Gambuzza, A. Cardillo, A. Fiasconaro, L. Fortuna, J. Gómez-Gardeñes, and M. Frasca, *Chaos* **23**, 043103 (2013).