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PARTITION THEORY AND THERMODYNAMICS OF MULTI-DIMENSIONAL OSCILLATOR ASSEMBLIES

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1. INTRODUCTION

The close similarity between the basic problems in statistical thermodynamics and the partition theory of numbers is now well recognized. In either case one is concerned with partitioning a large integer, under certain restrictions, which in effect means that the 'Zustandsumme' of a thermodynamic assembly is identical with the generating function of partitions appropriate to that assembly. The thermodynamic approach to the partition problem is of considerable interest as it has led to generalizations which so far have not yielded to the methods of the analytic theory of numbers. An interesting example is provided in a recent paper of Agarwala and Auluck (1) where the Hardy-Ramanujan formula for partitions into integral powers of integers is shown to be valid for non-integral powers as well.

The present paper is concerned with the problems in the partition theory of numbers corresponding to the thermodynamic assemblies of two and three-dimensional oscillators. Asymptotic expressions are deduced which constitute a generalization of the Hardy-Ramanujan (2) formula for $p(n)$ which corresponds to an assembly of *linear oscillators*. Generating functions similar to those considered here were studied earlier by MacMahon (3) in his work on combinatory analysis. It is remarkable that the Zustandsumme of an assembly of a variable number of two-dimensional oscillators is identical with the generating function of plane partitions. The problem, thus, becomes one of establishing a relationship between the two seemingly different types of partitions. Further, it is noticed that a study of two-dimensional oscillator assembly is connected with the partitions of bi-partite numbers.

The next two sections will be devoted to the case of the linear oscillator assembly and MacMahon's* approach to line, plane and solid partitions. The object is to provide an adequate background for generalizations to follow.

2. THE LINEAR OSCILLATOR ASSEMBLY AND LINE PARTITIONS†

Consider an assembly of N identical (non-interacting) linear harmonic oscillators. The energy levels of an oscillator in energy units are

$$\epsilon_j = j + \tfrac{1}{2} \quad (j = 0, 1, 2, \ldots).$$

If E denotes the total energy of the assembly, the energy actually available for distribution among the oscillators is

$$n = E - \tfrac{1}{2}N, \tag{1}$$

* I am grateful to the referee for drawing my attention to MacMahon's work.

† It might be noted that partitions studied by Hardy and Ramanujan and subsequently investigated by other authors belong to the class of line partitions.

where $\frac{1}{2}N$ denotes the zero point energy of the oscillators. If n_j represents the number of oscillators in the j state

$$n = \sum_j jn_j, \quad \sum n_j = N. \tag{2}$$

When the assembly obeys Bose-Einstein statistics n_j can have all integral positive values including zero. Equation (2) represents the partitions of n into integers with the only restriction that the total number of summands in no partition can exceed N. In the special case $N \geqslant n$ the second part of (2) becomes inoperative and the number of accessible wave functions of the assembly becomes equal to $p(n)$—the number of unrestricted partitions of n. It can be shown by the methods of the analytic theory of numbers and also by well-known formulae of statistical mechanics* that

$$p(n) \sim \frac{1}{4\sqrt{3}\,n} \exp\{\pi\sqrt{(\tfrac{2}{3}n)}\}. \tag{3}$$

It is usual to regard a partition as a collection of numbers whose sum is equal to the number partitioned. There is *a priori* no specification of order amongst the summands and thus any convenient order may be adopted. MacMahon has imported the idea of descending order of magnitude amongst the summands. Accordingly a summand is considered to have 'the attribute of position as well as of magnitude, the position being determined by relative magnitude.' In the present case, for example, the summands may be regarded as placed at points along a line

$$\bullet \geqslant \bullet \geqslant \bullet \geqslant \bullet \geqslant$$

where the symbol $\geqslant$ regulates the part magnitude at each point. We can also depict a partition graphically. Thus for example the graph

$$\begin{array}{l} \rightarrow x \\ \downarrow \\ y \end{array} \qquad \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}$$

depicts the partition $4+3+2+1$. The number of rows represents the number of summands and the number of dots in each row the part magnitude. The graphical representation of a partition and the idea of order amongst the summands marks a great step forward in partition theory. This enables us to visualize partitions in a plane and eventually pass on to the idea of solid partitions.

3. The generalization to plane and solid partitions

In the graphical representation of a line partition the nodes can be replaced by units without altering in any way its meaning. If, however, we replace the units by integers so that descending order of magnitude is in evidence along the x and y axis we arrive at what may be termed as a plane partition. Clearly, the unit graph which depicts a line partition represents also a plane partition in which the part magnitude does not exceed unity. The rigorous derivation of the generating function of plane partitions is a lengthy process. It involves the idea of Diophantine inequalities and the use of

* See for example Auluck and Kothari (4).

lattice functions*. A short and intuitive method of arriving at the generating function is to assume

$$\frac{(l+1)^{s_1}}{(1)^{s_1}}\frac{(l+2)^{s_2}}{(2)^{s_2}}\frac{(l+3)^{s_3}}{(3)^{s_3}}\cdots \tag{4}$$

as the generating function for plane partitions in which the part magnitude does not exceed l (symbol (m) stands for $1-x^m$). Clearly for $l = 1$ the generating function (4) must reduce to the case of line partitions. This demands

$$s_1 = 1, \quad s_2 = 2, \quad s_3 = 3, \quad \ldots.$$

When there is no restriction on the part magnitude the generating function for plane partitions becomes

$$\frac{1}{(1)(2)^2(3)^3(4)^4\ldots}. \tag{5a}$$

We can also depict a plane partition graphically by replacing each part by a pile of nodes in the z direction. If we replace nodes by units the result is a solid partition in which the part magnitude is not greater than unity. MacMahon has not given a rigorous derivation of the generating function for solid partitions. But a simple reasoning as in the case of plane partitions leads to the generating function

$$\frac{1}{(1)(2)^3(3)^6\ldots(s)^{\frac{1}{2}(s^2+s)}\ldots} \tag{5b}$$

for solid partitions when there is no restriction on part magnitude.

4. Two-dimensional oscillator assembly and plane partitions

Here we consider the case when the number of oscillators in the assembly is not fixed. The Schrödinger equation for a two-dimensional oscillator in plane xy separates into x, y. The eigenfunctions are of the form

$$\psi = (\psi_{j_x})(\psi_{j_y}),$$

where ψ_{j_x} and ψ_{j_y} are the eigenfunctions for the linear case. The corresponding eigenvalues are

$$\begin{aligned} \epsilon_j &= (j_x+\tfrac{1}{2})+(j_y+\tfrac{1}{2}) \quad (j_x \text{ or } j_y = 0, 1, 2, \ldots), \\ &= j+1 \qquad\qquad\qquad\quad (j = 0, 1, 2, \ldots). \end{aligned}$$

Since, each value of j can be obtained by $(j+1)$ combinations of j_x and j_y the state ϵ_j has $j+1$ eigenfunctions corresponding to it. In other words the state of *energy* j is j-fold degenerate. The role played by these degenerate states in specifying the thermodynamic state of an assembly is brought out in the corresponding partition problem by regarding the integer j as capable of occurring in j different ways, namely $j_1, j_2, \ldots, j_j$. As the enumeration of partitions of n is essentially the same as finding the number of states accessible to the assembly when in the energy state n the replacement of j_i by j_k in any partition gives rise to a new partition although j_i and j_k have the same numerical value j. The difference between j_j and j_k has to be understood in this sense. To illustrate the departure from the linear case we enumerate the partitions of 3. In the one-dimensional case the partitions are:

$$3, \quad 2+1, \quad 1+1+1.$$

* An alternative method has been given by Chaundy (7).

In the two-dimensional case the number 3 can occur in three ways ($3_1, 3_2, 3_3$) the number 2 in two ways ($2_1, 2_2$) so that the different partitions of 3 are:

$$3_1, \quad 3_2, \quad 3_3, \quad 2_1+1, \quad 2_2+1, \quad 1+1+1.$$

The Zustandsumme of an assembly is as usual defined by

$$Z = \sum_{n=0}^{\infty} \omega(n) e^{-n\mu}, \tag{6}$$

where μ is the inverse of temperature in energy units and $\omega(n)$ the number of states accessible to the assembly when in the energy state n. Since in the present case the state of energy j is j-fold degenerate we can write

$$Z = \prod_{r=1}^{\infty} (1-e^{-\mu r})^{-r}. \tag{7}$$

This is identical with ($5a$) the generating function for the unrestricted plane partitions. On the other hand, (4) (the generating function for plane partitions in which the part magnitude does not exceed l) enumerates those $O^{(2)}$ partitions* in which the suffix of no summand exceeds l. The highest suffix that can be attached to any integer depends upon the maximum permissible degeneracy of the level which the integer represents. What is restricted to l in (4) is the degeneracy of levels higher than l. Degeneracy has thus the same role in $O^{(2)}$, partitions as the part magnitude in plane partitions. In fact according to the two methods of enumeration we can divide the partitions of n into n classes such that the sth class contains those $O^{(2)}$ partitions which have s as the highest suffix and plane partitions with s as the largest part magnitude. This results in an equal number of the two types of partitions in the same class. In Table 1 a classification of the partitions of 3 is shown. It has, however, not been possible to establish a one to one correspondence between $O^{(2)}$ and plane partitions.

Table 1

Partitions	Class I	Class II	Class III
$O^{(2)}$	3_1, 2_1+1, $1+1+1$	3_2, 2_2+1	3_3
Plane	1 1 1, 1 1, 1 1, 1 1	21, 2 1	3

We next proceed with the evaluation of the asymptotic formula. The expression for Z can be used to determine the various thermodynamic functions of the assembly. In particular

$$n = \left(-\frac{\partial}{\partial\mu}\log Z\right), \tag{8}$$

and

$$s = \mu n + \log Z. \tag{9}$$

* $O^{(s)}$ partitions signifies partitions appropriate to an assembly of s-dimensional oscillators.

Here s denotes the entropy of the assembly. An application of the Mellin-Burkill inversion formula to (6) gives

$$\omega(n) \sim \frac{e^{s}}{\left(-2\pi\frac{\partial n}{\partial\mu}\right)^{\frac{1}{2}}}, \tag{10}$$

which is known as Bethe's theorem.

If $p^{(2)}(n)$ denotes the number of partitions appropriate to the present assembly

$$\omega(n) = p^{(2)}(n).$$

Equations (7) and (8) give

$$n = \sum_{r=0}^{\infty}\frac{r^2}{e^{\mu r}-1}. \tag{11a}$$

To evaluate the sum, we replace the summation by integration by the help of Euler-Maclaurin formula and have, for $\mu \to 0$,

$$\begin{aligned} n &= \int_0^{\infty}\frac{x^2}{e^{\mu x}-1}dx - \frac{1}{12\mu} + O(\mu) \\ &= \frac{2\zeta(3)}{\mu^3} - \frac{1}{12\mu} + O(\mu), \end{aligned} \tag{11b}$$

where $\zeta(s)$ is the Riemann zeta-function. Integrating (11b) with respect to μ, we obtain

$$\log Z = \frac{\zeta(3)}{\mu^2} + \frac{1}{12}\log\mu + c + O(\mu^2), \tag{12}$$

where c is the constant of integration.

To evaluate the constant we write equation (7) in the form

$$\log Z = \sum_{r=1}^{N-1}\log(1-e^{-\mu r})^{-r} + \sum_{r=N}^{\infty}\log(1-e^{-\mu r})^{-r}, \tag{13}$$

where N is a very large number. Now

$$\begin{aligned} \sum_{r=1}^{N}\log(1-e^{-\mu r})^{-r} &= -\sum_{r=1}^{N} r(\log\mu + \log r) + O(\mu) \\ &= -\tfrac{1}{2}N(N+1)\log\mu - \sum_{r=1}^{N} r\log r + O(\mu). \end{aligned} \tag{14}$$

It might be noted that $\zeta(s)$ can be written in the form*

$$\zeta(s) = \lim_{N\to\infty}\left\{\sum_{m=1}^{N} m^{-s} - \frac{N^{1-s}}{1-s} - \tfrac{1}{2}N^{-s} + \tfrac{1}{12}N^{-s-1}\right\}, \tag{15}$$

for $\sigma > -3$ (σ is real part of s). A formal differentiation of (15) yields

$$\begin{aligned} \sum_{m=1}^{N} m^{-s}\log m - \frac{N^{1-s}\log N}{1-s} + \frac{N^{1-s}}{(1-s)^2} - \tfrac{1}{2}N^{-s}\log N \\ - \tfrac{1}{12}N^{-s-1} + s\tfrac{1}{12}N^{-s-1}\log N = \zeta'(s) + o(1), \end{aligned} \tag{16}$$

a result which can be verified by a procedure similar to that used by Hardy (5) for Stirling's formula. The term $o(1)$ here and later refers to the process $N \to \infty$. Putting

* This formula is not applicable in the neighbourhood of the pole at $s = 1$.

$s = -1$ in (16) we obtain the value of $\sum_1^N r \log r$. Thus equation (14) can now be written in the form

$$\sum_1^N \log(1-e^{-\mu r})^{-r} = -\tfrac{1}{2}N(N+1)\log\mu - \tfrac{1}{2}N^2\log N + \tfrac{1}{4}N^2 - \tfrac{1}{2}N\log N$$
$$-\tfrac{1}{12}\log N - \tfrac{1}{12} + \zeta'(-1) + O(\mu) + o(1). \tag{17}$$

Also,
$$-\int_0^N \log(1-e^{-\mu x})^{-x}\,dx = \tfrac{1}{2}N^2\log\mu + \tfrac{1}{2}N^2\log N - \tfrac{1}{4}N^2 + O(\mu). \tag{18}$$

Therefore

$$\sum_1^{N-1} \log(1-e^{-\mu r})^{-r} - \int_0^N \log(1-e^{-\mu x})^{-x}\,dx$$
$$= \tfrac{1}{2}N(\log\mu + \log N) - \tfrac{1}{12}\log N - \tfrac{1}{12} + \zeta'(-1) + O(\mu) + o(1). \tag{19}$$

For the second part in (13), using Euler-Maclaurin formula we have

$$\sum_N^\infty \log(1-e^{-\mu r})^{-r} - \int_N^\infty \log(1-e^{-\mu x})^{-x}\,dx$$
$$= -\tfrac{1}{2}N(\log\mu + \log N) + \tfrac{1}{12}\{\log\mu + \log N + 1 + O(\mu)\} + O(1/N). \tag{20}$$

Adding (19) and (20) and letting $N \to \infty$, we obtain

$$\sum_{r=1}^\infty \log(1-e^{-\mu r})^{-r} - \int_0^\infty \log(1-e^{-\mu x})^{-x}\,dx = \tfrac{1}{12}\log\mu + \zeta'(-1) + O(1/N). \tag{21}$$

A comparison of (12) and (21) gives the value of the constant as $\zeta'(-1)$. Now the entropy

$$s = n\mu + \log Z$$
$$= 3\zeta(3)/\mu^2 - \tfrac{1}{12} + \tfrac{1}{12}\log\mu + c + O(\mu^2). \tag{22}$$

Differentiating (11a) with respect to μ and evaluating the sums, we obtain

$$-\frac{\partial n}{\partial \mu} = \frac{6\zeta(3)}{\mu^4} + O\left(\frac{1}{\mu^2}\right). \tag{23}$$

From (11b),
$$\mu \sim \left(\frac{2\zeta(3)}{n}\right)^{\frac{1}{3}} - \frac{1}{36n}. \tag{24}$$

Substituting (22), (23) and (24) in equation (10), we have after some simplification*

$$p^{(2)}(n) \sim (12\pi\zeta(3))^{-\frac{1}{2}}\,(n')^{-25/36}\exp\{3\zeta(3)\,n'^{\frac{2}{3}} + \zeta'(-1)\}, \tag{25}$$

where
$$n' = n/2\zeta(3).$$

5. The two-dimensional oscillator assembly and partitions of bipartite numbers

Here we consider the case when the number of oscillators in the assembly is a fixed number N. If E denotes the energy of the assembly, the actual energy available for sharing among the oscillators is
$$n = E - N. \tag{26}$$

* This result has also been proved by Wright(8) who gives additional terms of smaller order inside the exponential. It has been thought worth while to give another proof as the method used is applied later when we consider solid partitions. See also Brigham (9).

Equation (2) as before describes the partitions of n where each j now is assumed to be capable of occurring in $j+1$ different ways. Denote the number of partitions of n into N or less parts by $R_N^{(2)}(n)$. Then for $N \geqslant n$ the second part of (2) becomes inoperative and we have $R_N^{(2)}(n) = R^{(2)}(n)$ the number of unrestricted partitions of n. Thus the Zustandsumme

$$Z = \sum_n R^{(2)}(n)\, e^{-n\mu} = \prod_{r=1}^{\infty} (1-e^{-r\mu})^{-(r+1)} \quad (\text{for } N \geqslant n), \tag{27}$$

or

$$\log Z = \sum_{r=1}^{\infty} \log (1-e^{-r\mu})^{-r} + \sum_{r=1}^{\infty} \log (1-e^{-r\mu})^{-1}.$$

The first part on the right has already been evaluated in the previous section; for the second part we employ the functional relation

$$f(\mu) = \sqrt{\left(\frac{\mu}{2\pi}\right)} \exp\left\{\frac{\pi^2}{6\mu} - \frac{\mu}{24}\right\} f(\mu'), \tag{28}$$

where

$$f(\mu) = \prod_{r=1}^{\infty} (1-e^{-r\mu})^{-1} \text{ and } \mu' = 4\pi^2/\mu.$$

Finally, $\log Z = \{\zeta(3)/\mu^2 + \tfrac{1}{12}\log\mu + c + O(\mu^2)\}$

$$+ \{\zeta(2)/\mu + \tfrac{1}{2}\log\mu - \tfrac{1}{2}\log 2\pi + O(\mu)\}. \tag{29}$$

As before, differentiating $\log Z$ with respect to μ, we find

$$n = \sum_{r=1}^{\infty} \frac{r(r+1)}{e^{\mu r}-1}, \tag{30a}$$

which on simplification gives

$$\mu n = \frac{2\zeta(3)}{\mu^2} + \frac{\zeta(2)}{\mu} - \tfrac{7}{12} + O(\mu). \tag{30b}$$

From (30a)

$$-\frac{\partial n}{\partial \mu} = \frac{6\zeta(3)}{\mu^4} + O\left(\frac{1}{\mu^3}\right). \tag{31}$$

Inverting (30b), we obtain

$$\frac{1}{\mu} \sim \alpha n^{\frac{1}{3}} + \beta + \gamma n^{-\frac{1}{3}}, \tag{32}$$

where $\alpha = \{2\zeta(3)\}^{-\frac{1}{3}}, \quad \beta = -\left\{\dfrac{\zeta(2)\,\alpha^3}{3}\right\}, \quad \gamma = \dfrac{[\zeta(2)]^2\,\alpha^5}{9} + \tfrac{7}{36}\alpha^2.$

Adding (29) and (30b), we get

$$s = \frac{3\zeta(3)}{\mu^2} + \frac{2\zeta(2)}{\mu} + \tfrac{7}{12}\log\mu + (c - \tfrac{1}{2}\log 2\pi - \tfrac{7}{12}) + O(\mu). \tag{33}$$

Substitution of (31), (32) and (33) in (10) gives

$$R^{(2)}(n) \sim \frac{[6\zeta(3)]^{-\frac{1}{2}}}{2\pi n'^{31/36}} \exp\{3\zeta(3)\, n'^{\frac{2}{3}} + \zeta(2)\, n'^{\frac{1}{3}} + A\}, \tag{34}$$

where

$$A = \zeta'(-1) - \frac{[\zeta(2)]^2}{12\zeta(3)}.$$

The expression for $R^{(2)}(n)$ is of interest from the point of view of partitions of bipartite numbers* known also as compound denumeration. If $p[(lm)]$ denotes the partition of a bipartite ((3), loc. cit.)

$$\sum_m \sum_l p[(lm)]\, x^l y^m = \frac{1}{(1-x)(1-y)(1-x^2)(1-xy)(1-y^2)\ldots}. \tag{35}$$

The problem of evaluating the coefficients in the general case bristles with difficulties. If we are interested in the total number of partitions of all bipartites satisfying $l+m = n$ we have merely to pick the coefficient of x^n in (35) after putting $x = y$. Hence,

$$\sum_n \sum_{m=0}^{n} p[(n-m,m)]\, x^n = \prod_{r=1}^{\infty} (1-x^r)^{-(r+1)}. \tag{36}$$

The right-hand side is identical with the Zustandsumme of the assembly under consideration. Therefore

$$\sum_{m=0}^{n} p[(n-m,m)] = R^{(2)}(n). \tag{37}$$

6. Three-dimensional oscillator assembly

We shall now give an asymptotic expression for $R^{(3)}(n)$ the number of unrestricted partitions appropriate to an assembly of a fixed number of three-dimensional oscillators. No restriction upon the number of summands requires that the number of oscillators N be greater than n the energy actually available for distribution among the oscillators. The Schrödinger equation gives the eigenvalues

$$\epsilon_j = j + \tfrac{3}{2} \quad (j = 0, 1, 2, \ldots) \tag{38}$$

and to each value of j there correspond $\frac{1}{2}(j+1)(j+2)$ independent wave functions. Therefore, the integer j in any partition has to be regarded as capable of occurring in $\frac{1}{2}(j+1)(j+2)$ different ways. If E is the total energy of the assembly

$$n = E - \tfrac{3}{2}N. \tag{39}$$

The Zustandsumme

$$Z = \sum_n R^{(3)}(n)\, e^{-n\mu} = \prod_{r=1}^{\infty} (1-e^{-r\mu})^{-\frac{1}{2}(r+1)(r+2)} \quad (N \geqslant n). \tag{40}$$

Define two quantities Z_1 and Z_2 by the relations

$$Z_1 = \prod_{r=1}^{\infty} (1-e^{-r\mu})^{-\frac{1}{2}r(r+1)}, \tag{41}$$

$$Z_2 = \prod_{r=1}^{\infty} (1-e^{-r\mu})^{-(r+1)}, \tag{42}$$

where

$$\log Z = \log Z_1 + \log Z_2. \tag{43}$$

Since Z_2 has been evaluated in the previous section we have merely to evaluate Z_1. Incidentally, this will lead us very close to the asymptotic formula for solid partitions as Z_1 is identical with ($5b$).

* It may be noted that (lm) is a symbolic representation of complex quantity $l + im$. The problem of partitions of complex numbers will be taken up in detail in another paper to be published elsewhere.

Differentiating (41) with respect to μ, and after evaluating the sums, we have

$$\mu n_1 = \frac{3\zeta(4)}{\mu^3} + \frac{\zeta(3)}{\mu^2} - \tfrac{1}{24} + O(\mu). \tag{44}$$

Integrating, we obtain

$$\log Z_1 = \frac{\zeta(4)}{\mu^3} + \frac{\zeta(3)}{2\mu^2} + \tfrac{1}{24}\log\mu + c' + O(\mu). \tag{45}$$

The constant of integration c' is found to be $\frac{1}{2}\{\zeta'(-1)+\zeta'(-2)\}$ by the procedure of the previous section. Thus, finally

$$\log Z = \frac{\zeta(4)}{\mu^3} + \frac{3}{2}\frac{\zeta(3)}{\mu^2} + \frac{\zeta(2)}{\mu} + \tfrac{5}{8}\log\mu - \tfrac{1}{2}\log 2\pi + c + c' + O(\mu) \tag{46}$$

and

$$\mu n = \frac{3\zeta(4)}{\mu^3} + \frac{3\zeta(3)}{\mu^2} + \frac{\zeta(2)}{\mu} - \tfrac{5}{8} + O(\mu). \tag{47}$$

Therefore

$$s = \frac{4\zeta(4)}{\mu^3} + \frac{9}{2}\frac{\zeta(3)}{\mu^2} + \frac{2\zeta(2)}{\mu} + \tfrac{5}{8}(\log\mu - 1) - \tfrac{1}{2}\log 2\pi + c + c' + O(\mu). \tag{48}$$

Inverting (47), we obtain

$$\mu^{-1} = \alpha_1 n^{\frac{1}{4}} + \beta_1 + \gamma_1 n^{-\frac{1}{4}} + \delta_1 n^{-\frac{1}{2}} + O(n^{-\frac{3}{4}}), \tag{49}$$

where

$$\alpha_1 = \{3\zeta(4)\}^{-\frac{1}{4}}, \quad \beta_1 = -\tfrac{3}{4}\zeta(3)\,\alpha_1^4,$$

$$\gamma_1 = \tfrac{27}{32}\alpha_1^7[\zeta(3)]^2 - \tfrac{1}{4}\alpha_1^3\zeta(2),$$

$$\delta_1 = -\tfrac{27}{32}\alpha_1^{10}[\zeta(3)]^3 + \tfrac{3}{8}\zeta(3)\,\zeta(2)\,\alpha_1^6 + \tfrac{5}{32}\alpha_1^2.$$

Also

$$-\frac{\partial n}{\partial\mu} = \frac{12\zeta(4)}{\mu^5} + O\left(\frac{1}{\mu^4}\right). \tag{50}$$

Substituting (48), (49) and (50) in (10), we find

$$R^{(3)}(n) \sim \frac{\{3\zeta(4)\}^{-\frac{1}{2}}}{4\pi n'^{25/32}} \exp\left\{4\zeta(4)\,n''^{\frac{3}{4}} + \tfrac{3}{2}\zeta(3)\,n''^{\frac{1}{2}} + \left(\zeta(2) - \frac{3[\zeta(3)]^2}{8\zeta(4)}\right)n''^{\frac{1}{4}} + B_1\right\}, \tag{51}$$

where

$$B_1 = \frac{[\zeta(3)]^3}{8[\zeta(4)]^2} - \frac{\zeta(3)\,\zeta(2)}{4\zeta(4)} + \tfrac{3}{2}\zeta'(-1) + \tfrac{1}{2}\zeta'(-2),\ n'' = n/3\zeta(4).$$

On the other hand if we start with (41) the generating function for MacMahon's solid partitions (whose number we denote by $p^{(3)}(n)$) and follow the procedure, the details of which have been given twice, we get

$$p^{(3)}(n) \sim \frac{(n'')^{-61/96}}{(24\pi\zeta(4))^{\frac{1}{2}}} \exp\left\{4\zeta(4)\,n''^{\frac{3}{4}} + \tfrac{1}{2}\zeta(3)\,n''^{\frac{1}{2}} - \frac{(\zeta(3))^2}{24\zeta(4)}n''^{\frac{1}{4}} + \tfrac{1}{2}B_2\right\}, \tag{52}$$

where

$$B_2 = \frac{[\zeta(3)]^3}{108[\zeta(4)]^2} + \zeta'(-1) + \zeta'(-2).$$

7. Comparison of calculated and enumerated values

In Table 2 the calculated and the enumerated values of $p^{(2)}(n)$ and $p^{(3)}(n)$ are given for n up to 25 (at intervals of 5). The enumerations can be carried out by the help of recurrence relations given below.

Replacing $e^{-\mu}$ by x in equation (7), we find

$$\log Z = -\sum_{r=1}^{\infty} r \log(1-x^r) = \log \sum_{n=0}^{\infty} p^{(2)}(n)\, x^n. \tag{53}$$

Differentiating with respect to x, we obtain

$$\frac{Z'x}{Z} = \sum_{r=1}^{\infty} \frac{r^2 x^r}{1-x^r} = \sum_{n=0}^{\infty} np^{(2)}(n)\, x^n \Big/ \sum_{n=0}^{\infty} p^{(2)}(n)\, x^n. \tag{54}$$

On comparing the coefficients of x^n on both sides of (54), the relation

$$np^{(2)}(n) = \sum_{m=1}^{n} \sigma^{(2)}(m)\, p^{(2)}(n-m), \tag{55}$$

follows immediately. Similarly, it can be shown that

$$2np^{(3)}(n) = \sum_{m=1}^{n} (\sigma^{(2)}(m) + \sigma^{(3)}(m))\, p^{(3)}(n-m), \tag{56}$$

where $\sigma^{(r)}(m)$ denotes the sum of the rth powers of the divisors of m.

Table 2

n	$p^{(2)}(n)$		$p^{(3)}(n)$	
	Calculated	Enumerated	Calculated	Enumerated
5	27	24	62	59
10	526	500	3167	3162
15	7124	6879	110355	110445
20	77827	75278	2979182	2992892
25	716468	696033	66964233	67405569

We now break the partitions into classes such that partitions having a particular integer as the smallest summand* are placed in the same class. Thus, if $p^{(2)}(n, m)$ denotes the number of partitions of n having m as the smallest summand we notice at once that

$$p^{(2)}(n, 1) = \sum_{m=1}^{n-1} p^{(2)}(n-1, m) \equiv p^{(2)}(n-1),$$

$$p^{(2)}(n, m) = 0 \quad \text{for} \quad \tfrac{1}{2}n < m < n,$$

$$p^{(2)}(n, m) = n \quad \text{for} \quad n = m.$$

In fact, for a general m it can be shown from elementary considerations that

$$p^{(s)}(n, m) = \sum_{r=1}^{m} (-1)^{r+1} \binom{s(m)}{m} V^{(s)}(n-rm, m), \tag{57}$$

where $\quad s(m) = m \quad \text{for} \quad s = 2, \qquad s(m) = \tfrac{1}{2}m(m+1) \quad \text{for} \quad s = 3,$

and $$V^{(s)}(n, m) = \sum_{t=n}^{n} p^{(s)}(n, t).$$

The values of $p^{(2)}(n)$ and $p^{(3)}(n)$ obtained from (57) serve as a check on the values obtained from (55) and (56).

* The value of the summand here refers to the $O^{(s)}$ partitions and is not to be confused with part magnitude in MacMahon's plane and solid partitions.

For the calculation of $p^{(2)}(n)$ and $p^{(3)}(n)$ from the asymptotic formulae (25) and (52) we require the numerical value of $\zeta'(-1)$ and $\zeta'(-2)$. Dwight (6) has tabulated the values of $\zeta'(s)/\zeta(s)$. Since $\zeta(-2) = 0$ we cannot obtain $\zeta'(-2)$ from these tables. But, from the functional relation

$$\zeta(1-s) = \{2/(2\pi)^s\} \cos \tfrac{1}{2}\pi s \, \Gamma(s) \, \zeta(s) \tag{58}$$

we have, by differentiation, for $s = 3$

$$\zeta'(-2) = \zeta(3)/(2\pi^2).$$

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