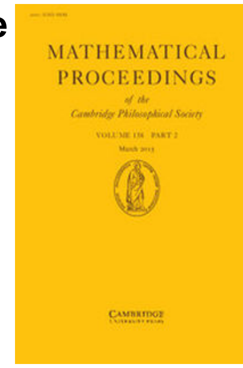

Asymptotic formulae in the theory of partitions

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ASYMPTOTIC FORMULAE IN THE THEORY OF PARTITIONS

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1. *Introduction.* It is the object of this paper to obtain an asymptotic formula for the number of partitions $p_m(n)$ of a large positive integer n into m parts λ_r, where the number m becomes large with n and the numbers $\lambda_1, \lambda_2, \ldots$ form a sequence of positive integers. The formula is proved by using the classical method of contour integration due to Hardy, Ramanujan and Littlewood. It will be necessary to assume certain conditions on the sequence λ_r, but these conditions are satisfied in most of the cases of interest. In particular, we shall be able to prove the asymptotic formula in the cases of partitions into positive integers, primes and kth powers for any positive integer k.

By considering the differences of the function $p_m(n)$ with respect to the integral variables m and n we shall prove (at any rate for sufficiently large n) a conjecture of Auluck, Chowla and Gupta (1), who predicted that, in the case of partitions into the positive integers, the function $p_m(n)$, when regarded as a function of m, would attain its maximum value for at most two consecutive values of m. We shall deduce from our asymptotic formulae that in a certain range of m (depending on n) the function $p_m(n)$ increases monotonically with m until it reaches its maximum value and then decreases monotonically, and that this maximum is not attained for any m outside the range.

The asymptotic formulae that we prove have some applications to the theory of Bose-Einstein assemblies in statistical mechanics; it is hoped to publish an account of these applications in due course.

Ingham (5) has developed a method of Tauberian analysis which enables us to obtain an asymptotic formula for the total number of partitions $p(n)$ of n into parts λ_r, assuming only some rather weak conditions on the λ_r. Ingham shows how asymptotic formulae may be obtained for the number of numbers in a given range which can be partitioned into integral or non-integral parts λ_r, weighted according to the number of partitions. If the number of solutions of the inequality

$$r_1\lambda_1 + r_2\lambda_2 + \ldots < u \tag{1}$$

in integers $r_i \geqslant 0$ is denoted by $P(u)$, Ingham obtains an asymptotic formula for

$$P_h(u) = \frac{P(u) - P(u-h)}{h},$$

which in the case of partitions into integers reduces to the number of partitions of the largest integer less than u, when $h = 1$. The results of several recent papers may be obtained immediately by using Ingham's Theorem 1, or more simply in some of the easier cases by using Theorem 2. Auluck and Haselgrove (2) have shown how some of Ingham's conditions may be relaxed. For the sake of simplicity we shall not extend our results to partitions into non-integral parts in the present paper, but it is fairly simple to carry through the necessary analysis, and the results would be very similar to those of this paper.

We do not use Ingham's Theorem in the theory below as we have assumed different conditions on the λ_r in order to prove the asymptotic formulae for the number of partitions of n into m parts. As the formulae for the total number of partitions follow from our results, there is no need to use Ingham's method, which would make it necessary to prove further properties of the λ_r (although these can probably be deduced from our assumptions).

We shall use the symbols O, o and $\sim$; the constants involved in the O symbol will be independent of the variables, but may depend on certain quantities which we may regard temporarily as fixed. Similarly, the symbols o and $\sim$ will hold uniformly in the variables.

2. *The generating function.* We see that $p_m(n)$ is the number of solutions in non-negative integers r_i of the equations

$$\left.\begin{aligned} r_1\lambda_1+r_2\lambda_2+\ldots &= n,\\ r_1+r_2+\ldots &= m.\end{aligned}\right\} \tag{2}$$

The generating function of $p_m(n)$ is

$$\begin{aligned} G(x,z) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_m(n)\,x^m z^n \\ &= \prod_{r=1}^{\infty}(1-xz^{\lambda_r})^{-1}, \end{aligned}\tag{3}$$

where $0<\lambda_1\leqslant\lambda_2\leqslant\lambda_3\leqslant\ldots$ is an infinite sequence of positive integers. We shall assume that the series

$$\sum_r \lambda_r^{-2} \tag{4}$$

converges. If this series diverges it is possible, as we shall indicate at the end of § 3, to apply the saddle-point method to both of the variables x and z (as is often done in works on statistical mechanics), provided only that suitable conditions on the λ_r are satisfied. But when the series (4) converges the saddle-point method cannot be used for the x variable (a formal application leads to an incorrect asymptotic formula for $p_m(n)$), and a different method such as that introduced in this paper must be used.

It is convenient to make some changes of variable. We write $x = e^{-v}$, $z = e^{-\omega}$ and then $v = \alpha\omega$. Then if $G(x,z) = g(\alpha,\omega)$

$$g(\alpha,\omega) = \prod_{r=1}^{\infty}\{1-e^{-(\lambda_r+\alpha)\omega}\}^{-1}. \tag{5}$$

If $\Re\omega>0$ the product for $g(\alpha,\omega)$ is absolutely convergent. Hence $g(\alpha,\omega)$ is an analytic function of α with poles at the points $\alpha = -\lambda_r = 2s\pi i\omega^{-1}$ for any fixed value of ω. We shall be concerned, in the first instance, with the behaviour of $g(\alpha,\omega)$ for small ω. We construct a function with similar behaviour.

Let

$$K(\alpha) = \prod_{r=1}^{\infty}\left\{1+\frac{\alpha}{\lambda_r}\right\}^{-1} e^{\alpha/\lambda_r}. \tag{6}$$

Since $\Sigma\lambda_r^{-2}$ converges, the infinite product for $K(\alpha)$ converges, and $K(\alpha)$ is an analytic function with poles at the points $\alpha = -\lambda_r$. Thus we see that $g(\alpha,\omega)/K(\alpha)$ is a regular function of α for $|\alpha|<2\pi\Re\omega^{-1}$. Further, this function has no zeros. Hence

$$\chi(\alpha,\omega) = \log\{K(\alpha)/g(\alpha,\omega)\}$$

is regular for $|\alpha|<2\pi\Re\omega^{-1}$.

We shall now prove the

LEMMA. *If $\omega \to 0$ in a fixed Stolz angle Δ (i.e. $\omega \to 0$ in such a way that $|\Im\omega| < \Delta\Re\omega$), and if $|\alpha| \leqslant \frac{3}{2}\pi/|\omega|\sqrt{(1+\Delta^2)}$, then*

$$\chi(\alpha, \omega) = o(|\omega|^{-2}). \tag{7}$$

To prove this lemma we observe that

$$\Re\chi(\alpha, \omega) = \sum_{r=1}^{\infty} \log \left| \frac{1 - e^{-(\lambda_r + \alpha)\omega}}{\left(1 + \frac{\alpha}{\lambda_r}\right) e^{-\alpha/\lambda_r}} \right|.$$

Now for $|\alpha| = 8\pi|\omega|^{-1}$ and $\lambda_r \leqslant 4\pi|\omega|^{-1}$,

$$\left| \frac{1 - e^{-(\lambda_r + \alpha)\omega}}{\left(1 + \frac{\alpha}{\lambda_r}\right) e^{-\alpha/\lambda_r}} \right| \leqslant (1 + e^{12\pi})\, e^{8\pi/(\lambda_r|\omega|)}.$$

By the maximum modulus principle applied to the complex variable α the above inequality holds for $|\alpha| \leqslant 8\pi|\omega|^{-1}$ and $\lambda_r \leqslant 4\pi|\omega|^{-1}$.

Now for $|\alpha| \leqslant 2\pi|\omega|^{-1}$, $\quad |e^{-(\lambda_r+\alpha)\omega}| \leqslant e^{2\pi - \lambda_r\Re\omega}$.

If also $\lambda_r > 4\pi|\omega|^{-1}$ we have

$$\left(1 + \frac{\alpha}{\lambda_r}\right) e^{-\alpha/\lambda_r} = 1 + O\left(\frac{|\alpha|^2}{\lambda_r^2}\right).$$

Hence for $|\alpha| \leqslant 2\pi|\omega|^{-1}$,

$$\Re\chi(\alpha, \omega) \leqslant \sum_{\lambda_r \leqslant 4\pi|\omega|^{-1}} O\left(\frac{|\omega|^{-1}}{\lambda_r}\right) + \sum_{\lambda_r > 4\pi|\omega|^{-1}} O\left(e^{-\lambda_r\Re\omega} + \frac{|\alpha|^2}{\lambda_r^2}\right).$$

Since $\Sigma\lambda_r^{-2}$ converges,

$$\sum_{\lambda_r \leqslant 4\pi|\omega|^{-1}} \frac{1}{\lambda_r} = o(|\omega|^{-1})$$

and

$$\sum_{\lambda_r > 4\pi|\omega|^{-1}} e^{-\lambda_r\Re\omega} = \sum_{\lambda_r > 4\pi|\omega|^{-1}} O\left(\frac{1}{(\lambda_r\Re\omega)^2}\right) = o((\Re\omega)^{-2}).$$

Hence for $|\alpha| \leqslant 2\pi|\omega|^{-1}$, $\quad \Re\chi(\alpha, \omega) \leqslant o((\Re\omega)^{-2})$.

It should be noted that we have not proved that $\Re\chi(\alpha, \omega)$ cannot take arbitrarily large negative values. In fact it will take such values in the neighbourhood of the poles of $g(\alpha, \omega)$ which are not poles of $K(\alpha)$.

Now

$$\begin{aligned}\chi(0, \omega) &= \sum_1^{\infty} \log(1 - e^{-\lambda_r\omega}) \\ &= \sum_{\lambda_r \leqslant 4\pi|\omega|^{-1}} O\left(\frac{1}{\lambda_r|\omega|}\right) + \sum_{\lambda_r > 4\pi|\omega|^{-1}} O(e^{-\lambda_r\Re\omega}) \\ &= o((\Re\omega)^{-2}).\end{aligned}$$

But $\chi(\alpha, \omega)$ is regular for $|\alpha| < 2\pi\Re\omega^{-1}$, and so for $|\alpha| < 2\pi/|\omega|\sqrt{(1+\Delta^2)}$, so that we deduce, using Carathéodory's theorem, that for $|\alpha| \leqslant \frac{3}{2}\pi/|\omega|\sqrt{(1+\Delta^2)}$

$$\chi(\alpha, \omega) = o((\Re\omega)^{-2}),$$

which is the required result.

We may now apply Cauchy's integral formula to deduce an estimate for

$$\frac{\partial^2}{\partial\alpha^2}\chi(\alpha, \omega) = \frac{1}{2\pi i}\int \frac{\chi(\tau, \omega)}{(\tau - \alpha)^3}\, d\tau.$$

Integrating round a circle of radius $\frac{3}{2}\pi/|\omega|\sqrt{(1+\Delta^2)}$ and centre $\tau = 0$, we deduce that for $|\alpha| \leqslant \pi/|\omega|\sqrt{(1+\Delta^2)}$

$$\frac{\partial^2}{\partial\alpha^2}\chi(\alpha,\omega) = o(1). \tag{8}$$

Hence $$\chi(\alpha,\omega) = A(\omega)+\alpha B(\omega)+o(\alpha^2)$$

for $|\alpha| \leqslant \pi/|\omega|\sqrt{(1+\Delta^2)}$. Putting $\alpha = 0$ we deduce that

$$A(\omega) = -\log g(0,\omega).$$

Differentiating $\chi(\alpha,\omega)$ with respect to α at $\alpha = 0$ we deduce that

$$B(\omega) = \sum_{r=1}^{\infty}\frac{\omega}{e^{\lambda_r\omega}-1}.$$

We write for convenience $$B(\omega) = \omega m_0(\omega),$$

where $$m_0(\omega) = \sum_{r=1}^{\infty}\frac{1}{e^{\lambda_r\omega}-1}.$$

Hence $$g(\alpha,\omega) = K(\alpha)\,g(0,\omega)\exp\{-\alpha\omega m_0(\omega)+o(\alpha^2)\} \tag{9}$$

uniformly for $|\alpha| \leqslant \pi/|\omega|\sqrt{(1+\Delta^2)}$ as $\omega \to 0$ in Δ. It should be noted that the existence of poles of $K(\alpha)$ and $g(\alpha,\omega)$ does not affect this result.

The equation (9) separates the behaviour of $g(\alpha,\omega)$ between the variables α and ω. It will be shown that the variation of $m_0(\omega)$ can be neglected in our estimation of the integrals.

For shortness we write $g(0,\omega) = g(\omega)$ and then

$$\log g(\omega) = -\sum_1^{\infty}\log(1-e^{-\lambda_r\omega}) = \Psi(\omega).$$

Then $$\Psi'(\omega) = -\sum_1^{\infty}\frac{\lambda_r}{e^{\lambda_r\omega}-1},$$

$$\Psi''(\omega) = \sum_1^{\infty}\frac{\lambda_r^2\,e^{\lambda_r\omega}}{(e^{\lambda_r\omega}-1)^2}.$$

Alternatively we see that $$\Psi'(\omega) = -\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\lambda_r\,e^{-t\lambda_r\omega}.$$

Thus $$\Psi'''(\omega) = -\sum_{r=1}^{\infty}\sum_{t=1}^{\infty}\lambda_r^3 t^2\,e^{-t\lambda_r\omega}.$$

Hence if we split ω into its real and imaginary parts so that $\omega = \xi+i\eta$, we have

$$|\Psi'''(\omega)| \leqslant -\Psi'''(\xi). \tag{10}$$

We shall suppose that the λ_r are such that

$$\xi\Psi'''(\xi) = O\{\Psi''(\xi)\} \quad \text{for} \quad \xi \to 0. \tag{11}$$

Then $$|\Psi''(\omega)-\Psi''(\xi)| = O\left(\frac{\eta}{\xi}\Psi''(\xi)\right).$$

We then obtain the formula

$$\Psi(\omega) = \Psi(\xi)+i\eta\Psi'(\xi)-\tfrac{1}{2}\eta^2\Psi''(\xi)+O\left(\frac{\eta^3}{\xi}\Psi''(\xi)\right). \tag{12}$$

Further we see that $$\xi^2\Psi''(\xi) \to \infty \quad \text{as} \quad \xi \to 0. \tag{13}$$

Now
$$-m_0'(\xi) = \sum_1^\infty \frac{\lambda_r e^{\lambda_r\xi}}{(e^{\lambda_r\xi}-1)^2}$$
$$= \sum_1^\nu O(\xi^{-2}) + \sum_{\nu+1}^\infty \frac{\lambda_r e^{\lambda_r\xi}}{(e^{\lambda_r\xi}-1)^2}$$

for any integer ν such that $\lambda_\nu < 1/\xi$. Thus by Cauchy's inequality

$$m_0'(\xi) = O(\nu\xi^{-2}) + O\left\{\left(\sum_{\nu+1}^\infty \frac{\lambda_r^2 e^{\lambda_r\xi}}{(e^{\lambda_r\xi}-1)^2} \sum_{\nu+1}^\infty \frac{e^{\lambda_r\xi}}{(e^{\lambda_r\xi}-1)^2}\right)^{\frac{1}{2}}\right\}$$
$$= O(\nu\xi^{-2}) + O\left\{(\Psi''(\xi))^{\frac{1}{2}} \xi^{-1}\left(\sum_{\nu+1}^\infty \lambda_r^{-2}\right)^{\frac{1}{2}}\right\}.$$

Now since $\xi^2\Psi''(\xi) \to \infty$ and $\Sigma\lambda_r^{-2}$ converges, the right-hand side may be made $o\{\xi^{-1}(\Psi''(\xi))^{\frac{1}{2}}\}$ by making $\nu \to \infty$ in such a way that $\nu = o\{\xi(\Psi''(\xi))^{\frac{1}{2}}\}$ and $\lambda_\nu < 1/\xi$. Thus we obtain, since $|m_0'(\omega)| \leqslant -m_0'(\xi)$,

$$m_0(\omega) = m_0(\xi) + o(\eta\xi^{-1}\{\Psi''(\xi)\}^{\frac{1}{2}})$$

and
$$\omega m_0(\omega) = \omega m_0(\xi) + o(\eta\{\Psi''(\xi)\}^{\frac{1}{2}}), \tag{14}$$

as $\omega \to 0$ in Δ. Substituting these results in our formula (9) for $g(\alpha, \omega)$, we obtain

$$g(\alpha,\omega) = K(\alpha)\exp\{\Psi(\xi) + i\eta\Psi'(\xi) - \tfrac{1}{2}\eta^2\Psi''(\xi) - \alpha\omega m_0(\xi)\}$$
$$\exp\{o(\eta\alpha(\Psi''(\xi))^{\frac{1}{2}}) + O(\xi^{-1}\eta^3\Psi''(\xi)) + o(\alpha^2)\}, \tag{15}$$

provided that $|\eta| < \Delta\xi$ and $|\alpha| < \pi/\xi(1+\Delta^2)$ (since this last implies that
$$|\alpha| < \pi/|\omega|\sqrt{(1+\Delta^2)}).$$

3. *The contour integration.* The result (15) will enable us to estimate certain integrals involving $g(\alpha, \omega)$. Suppose, in the first instance, that we are dealing with the problem of partitions of an integer n into m integral parts of the type λ_r. To estimate this number, which we denote by $p_m(n)$, we use the relation

$$p_m(n) = \frac{1}{(2\pi i)^2}\iint \frac{G(x,z)}{x^{m+1}z^{n+1}}\,dx\,dz$$
$$= \frac{1}{(2\pi i)^2}\iint g(\alpha,\omega)\, e^{(m\alpha+n)\omega}\,\omega\, d\alpha\, d\omega. \tag{16}$$

We carry out the integration over the ranges $-i\pi+\xi$, $i\pi+\xi$ for ω, and $-i\pi/\omega$, $i\pi/\omega$ for α. If n is sufficiently large we may choose ξ so that $n + \Psi'(\xi) = 0$. We see that $\xi \to 0$ as $n \to \infty$. Then we split the ranges of integration into the parts

(*A*) $|\eta| < \mu\xi\delta$, $|\alpha| < \mu$,
(*B*) $|\eta| < \mu\xi\delta$, $\mu \leqslant |\alpha| \leqslant \pi|\omega|^{-1}$,
(*C*) $\mu\xi\delta \leqslant |\eta| \leqslant \pi$,

where $\delta = \xi^{-1}\{\Psi''(\xi)\}^{-\frac{1}{2}}$. We notice that $\delta \to 0$ as $n \to \infty$. We shall make $\mu \to \infty$ as $n \to \infty$, but later assume that $\mu \to \infty$ sufficiently slowly for certain conditions to be satisfied. We shall denote the contributions to $p_m(n)$ of the integrals over the ranges (*A*), (*B*) and (*C*) by J_A, J_B and J_C respectively.

Consider the range (*A*) first. We have to evaluate the integral

$$J_A = \frac{1}{(2\pi)^2 i}\int_{-\mu\xi\delta}^{\mu\xi\delta}\int_{-(i\mu|\omega|)/\omega}^{(i\mu|\omega|)/\omega} g(\alpha,\omega)\, e^{(m\alpha+n)\omega}\,\omega\, d\alpha\, d\eta. \tag{17}$$

Substituting for $g(\alpha,\omega)$ we see that

$$J_A = \frac{1}{(2\pi)^2 i}\int_{-\mu\xi\delta}^{\mu\xi\delta}\int_{-(i\mu|\omega|)/\omega}^{(i\mu|\omega|)/\omega} \exp\{\Psi(\xi)-\tfrac{1}{2}\eta^2\Psi''(\xi)+n\xi\}$$
$$\times K(\alpha)\exp\{(m-m_0(\xi))\,\omega\alpha\}\exp\{O(\eta^3\xi^{-1}\Psi''(\xi))+o(\eta\mu(\Psi''(\xi))^{\frac{1}{2}})+o(\mu^2)\}\,\omega\,d\alpha\,d\eta, \quad (18)$$

since if $\mu\to\infty$ sufficiently slowly there exists Δ such that $|\eta|<\Delta\xi$ and $|\alpha|\leqslant\pi/\xi(1+\Delta^2)$ throughout the range of integration. We easily see by the results above that all the error terms $O(\eta^3\xi^{-1}\Psi''(\xi))$, $o(\eta\mu(\Psi''(\xi))^{\frac{1}{2}})$ and $o(\mu^2)$ may be replaced by $o(1)$ throughout the range of integration provided that $\mu\to\infty$ sufficiently slowly.

We now observe that

$$K(\alpha) = O(|\alpha|^{-\Delta_1}) \quad \text{for any } \Delta_1>0, \quad (19)$$

as $|\alpha|\to\infty$ with $|\Re\alpha|<\frac{1}{2}$; for

$$\left|\frac{K(\alpha)}{K(\Re\alpha)}\right| = \prod_{r=1}^{\infty}\left|\frac{\lambda_r+\Re\alpha}{\lambda_r+\alpha}\right|$$

and

$$\left|\frac{\lambda_r+\Re\alpha}{\lambda_r+\alpha}\right|\leqslant 1.$$

We shall consider separately the cases

$$\text{(i)}\quad |m-m_0(\xi)|\,\xi\delta^{\frac{1}{2}}\leqslant 1,$$
$$\text{(ii)}\quad |m-m_0(\xi)|\,\xi\delta^{\frac{1}{2}}> 1.$$

In case (i) $$(m-m_0(\xi))\,\xi\delta\mu^2\to 0 \quad \text{if } \mu^4\delta\to 0,$$

which we may suppose to be the case, and then

$$\{m-m_0(\xi)\}\,\omega\alpha = \{m-m_0(\xi)\}\,\xi\alpha+o(1).$$

Thus (18) becomes

$$J_A = \frac{1}{(2\pi)^2 i}\int_{-\mu\xi\delta}^{\mu\xi\delta}\exp\{\Psi(\xi)+n\xi-\tfrac{1}{2}\eta^2\Psi''(\xi)\}$$
$$\times\int_{-(i\mu|\omega|)/\omega}^{(i\mu|\omega|)/\omega} K(\alpha)\exp\{(m-m_0(\xi))\,\xi\alpha+o(1)\}\,\omega\,dx\,d\eta. \quad (20)$$

Now $$\Re\left(\frac{i\mu|\omega|}{\omega}\right) = O(\mu^2\delta) = o(\delta^{\frac{1}{2}}), \quad (21)$$

since $\mu^4\delta = o(1)$. By (19) and (21)

$$\int_{-(i\mu|\omega|)/\omega}^{(i\mu|\omega|)/\omega} |K(\alpha)\exp\{(m-m_0(\xi))\,\xi\alpha\}\,d\alpha| = O(1),$$

and $$\int_{(i\mu|\omega|)/\omega}^{i\mu} |K(\alpha)\exp\{(m-m_0(\xi))\,\xi\alpha\}\,d\alpha| = o(1).$$

Further $$\int_{i\mu}^{i\infty} |K(\alpha)\exp\{(m-m_0(\xi))\,\xi\alpha\}\,d\alpha| = o(1).$$

Hence

$$\int_{-(i\mu|\omega|)/\omega}^{(i\mu|\omega|)/\omega} K(\alpha)\exp\{(m-m_0(\xi))\,\xi\alpha\}\,d\alpha = \int_{-i\infty}^{i\infty} K(\alpha)\exp\{(m-m_0(\xi))\,\xi\alpha\}\,d\alpha+o(1). \quad (22)$$

In the case (ii) we see that $|m-m_0(\xi)|\,|\omega|\to\infty$. Hence since the argument of the exponential is imaginary it can be proved by integration by parts that

$$\int_{-(i\mu|\omega|)/\omega}^{(i\mu|\omega|)/\omega} K(\alpha)\exp\{(m-m_0(\xi))\,\alpha\omega\}\,d\alpha = o(1) \quad (23)$$

for any fixed μ, using (19) and the fact, which can be deduced from (19) by Cauchy's integral formula, that $K'(\alpha) = O(|\alpha|^{-\Delta_1})$ in the strip $|\Re\alpha| < \frac{1}{4}$. The result (23) also holds if $\mu \to \infty$ sufficiently slowly. Also in this case by the Riemann-Lebesgue theorem

$$\int_{-i\infty}^{i\infty} K(\alpha) \exp\{(m - m_0(\xi))\,\xi\alpha\}\,d\alpha = o(1), \tag{24}$$

since

$$\int_{-i\infty}^{i\infty} |K(\alpha)\,d\alpha| < \infty. \tag{25}$$

Thus in both the cases (i) and (ii) we have

$$\int_{-(i\mu|\omega|)/\omega}^{(i\mu|\omega|)/\omega} K(\alpha) \exp\{(m - m_0(\xi))\,\alpha\omega\}\,d\alpha = \int_{-i\infty}^{i\infty} K(\alpha) \exp\{(m - m_0(\xi))\,\alpha\xi\}\,d\alpha + o(1). \tag{26}$$

By straightforward integration we obtain

$$\frac{1}{2\pi}\int_{-\mu\xi\delta}^{\mu\xi\delta} \exp\{\Psi(\xi) + n\xi - \tfrac{1}{2}\eta^2\Psi''(\xi)\}\,\omega\,d\eta$$
$$= \frac{1}{\sqrt{(2\pi)}}\,\xi\{\Psi''(\xi)\}^{-\frac{1}{2}} \exp[\Psi(\xi) + n\xi]\,\{1 + o(1)\}. \tag{27}$$

Thus

$$J_A = \frac{1}{\sqrt{(2\pi)}}\,\xi\{\Psi''(\xi)\}^{-\frac{1}{2}} \exp[\Psi(\xi) + n\xi] \left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K(\alpha) \exp\{(m - m_0(\xi))\,\alpha\xi\}\,d\alpha + o(1)\right). \tag{28}$$

We notice further that

$$\frac{1}{2\pi}\int_{-\mu\xi\delta}^{\mu\xi\delta} g(0, \omega)\,e^{n\omega}\,d\eta \sim \frac{1}{\sqrt{(2\pi)}}\{\Psi''(\xi)\}^{-\frac{1}{2}} \exp[\Psi(\xi) + m\xi]. \tag{29}$$

To deal with the ranges of integration (B) and (C) it is necessary to introduce some conditions on the numbers λ_r. These conditions are essential to the nature of the problem; for suppose that numbers λ_r were all even, then it would be impossible to partition an odd number into even parts and no smooth asymptotic formula would describe the behaviour of $p_m(n)$. Our proof of the asymptotic formula would then break down because the integrals over the range (C) could not be neglected.

We split the range (C) into the two parts

$$(C_1) \quad \mu\xi\delta < |\eta| \leqslant \xi\Delta,$$
$$(C_2) \quad \xi\Delta < |\eta| \leqslant \pi.$$

Δ will be supposed fixed but will ultimately be made sufficiently small. We write

$$\psi(\omega) = \sum_{1}^{\infty} e^{-\lambda_r\omega} \tag{30}$$

and suppose that, in the range (C_2),

$$|\psi(\omega)| \leqslant \theta\psi(\xi) \tag{31}$$

for all sufficiently small values of ξ, where θ is a positive number depending only on Δ and $\theta < 1$ for all Δ.

We further suppose that, as $\xi \to 0$,

$$\sum_{r=1}^{\infty} r^2\psi'''(r\xi) = O\{\psi'''(\xi)\}. \tag{32}$$

Now the same order relation holds for $\xi \to \infty$ so that it holds uniformly for all ξ. Hence by integration

$$\left.\begin{aligned} \sum_{r=1}^{\infty} r\psi''(r\xi) &= O\{\psi''(\xi)\}, \\ \sum_{r=1}^{\infty} \psi'(r\xi) &= O\{\psi'(\xi)\} \\ \text{and} \qquad \sum_{r=1}^{\infty} \frac{1}{r}\psi(r\xi) &= O\{\psi(\xi)\}. \end{aligned}\right\} \tag{33}$$

Thus, for small ξ,
$$\psi(\xi) > A \sum_{r<1/\xi} \frac{1}{r} > A \log\frac{1}{\xi}$$
and
$$\psi(\xi) > A \sum_{r<1/\xi} \frac{1}{r}\log\frac{1}{r\xi} > A\log^2\frac{1}{\xi}. \tag{34}$$

It is convenient to state the above conditions in terms of the function $\psi(\omega)$, as they then take their simplest forms. The second condition (32) is usually satisfied trivially, but (31) may be difficult to prove.

To deal with the range of integration (B) we observe that

$$G(x,z) = \prod_{r=1}^{\infty} (1-xz^{\lambda_r})^{-1}$$
and
$$|G(x,z)| = \prod_{r=1}^{\infty} \left|\frac{1-|xz^{\lambda_r}|}{1-xz^{\lambda_r}}\right| G(|x|,|z|).$$
Now
$$xz^{\lambda_r} = e^{-(\lambda_r+\alpha)\omega},$$
$$\arg xz^{\lambda_r} = -\Im(\lambda_r+\alpha)\,\omega = i\alpha\omega - \lambda_r\eta,$$
since $\alpha\omega$ is pure imaginary. Hence
$$|\arg xz^{\lambda_r}| \geqslant |\alpha|\,\xi - \lambda_r\mu\xi\delta = (|\alpha| - \lambda_r\mu\delta)\,\xi.$$
If we suppose that $\lambda_r < |\alpha|$ we obtain
$$|\arg xz^{\lambda_r}| > \tfrac{1}{2}|\alpha|\,\xi$$
for sufficiently small ξ, since $\mu\delta \to 0$. Also $|\alpha| \leqslant \pi\xi^{-1}$ so that $|xz^{\lambda_r}| \geqslant e^{-\pi}$ and
$$|\Im xz^{\lambda_r}| > A\,|\alpha|\,\xi.$$
But
$$\Re(1-xz^{\lambda_r}) \geqslant 1-|xz^{\lambda_r}| = 1-e^{-\lambda_r\xi}$$
and
$$1-e^{-\lambda_r\xi} = O(\lambda_r\xi) = O(|\alpha|\,\xi).$$
This suffices to prove that for $\lambda_r < |\alpha|$ we have
$$\left|\frac{1-|xz^{\lambda_r}|}{1-xz^{\lambda_r}}\right| < e^{-A}$$
for some $A > 0$, provided that ξ is sufficiently small. But
$$\left|\frac{1-|xz^{\lambda_r}|}{1-xz^{\lambda_r}}\right| \leqslant 1$$
always. Suppose $N(\lambda)$ is the number of $\lambda_r < \lambda$; then we have
$$|G(x,z)| < e^{-AN(|\alpha|)}G(|x|,|z|).$$

Then the integral over the range (B) is

$$J_B = O\left(\mu\xi^2\delta\,|\,z\,|^{-n}\,G(|\,x\,|,|\,z\,|)\int_\mu^\infty e^{-AN(u)}\,du\right)$$
$$= O\left\{\xi\{\Psi''(\xi)\}^{-\frac{1}{2}}\exp\left[\Psi(\xi)+n\xi\right]\mu\int_\mu^\infty e^{-AN(u)}\,du\right\}. \tag{35}$$

It follows from (4) and (34) that

$$N(u)/\log u \to \infty \quad \text{as} \quad u\to\infty,$$

which implies that $$J_B = o(\xi\{\Psi''(\xi)\}^{-\frac{1}{2}}\exp\left[\Psi(\xi)+n\xi\right]), \tag{36}$$
since $\mu\to\infty$ as $\xi\to 0$.

For the range (C_2) we have

$$\log G(x,z) = \sum_{r=1}^{\infty}\frac{1}{r}x^r\psi(r\omega),$$

so that $$|\log G(x,z)\,| \leqslant |\,\psi(\omega)\,| + \sum_2^\infty \frac{1}{r}\,|\,\psi(r\omega)\,|$$
$$\leqslant \theta\psi(\xi) + \sum_2^\infty \frac{1}{r}\,\psi(r\xi)$$

by (31). (It is necessary to split the sum as (31) holds only for sufficiently small ξ.) Then using (33) we obtain

$$|\log G(x,z)\,| \leqslant \theta'\Psi(\xi), \quad \text{where} \quad \theta' < 1.$$

Further $$\Psi(\xi) \geqslant \psi(\xi) \geqslant A\log^2\frac{1}{\xi}$$

for some positive constant A. Thus in the range (C_2)

$$|\,G(x,z)\,| = O\left(\exp\left[\Psi(\xi) - A\log^2\frac{1}{\xi}\right]\right).$$

Then it follows that

$$|\,J_{C_2}| \leqslant \frac{1}{4\pi^2}\iint_{C_2}\left|\frac{G(x,z)}{x^{m+1}z^{n+1}}\,dx\,dz\right| = O\left(\exp\left[\Psi(\xi)+n\xi - A\log^2\frac{1}{\xi}\right]\right). \tag{37}$$

To deal with the range (C_1) we use the relation

$$|\,G(x,z)\,| = \left|\prod_{r=1}^{\infty}(1-xz^{\lambda_r})^{-\frac{1}{2}}\exp\left[\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r}x^r\psi(r\omega)\right]\right|$$
$$\leqslant \prod_{r=1}^{\infty}\left|\frac{1-|\,xz^{\lambda_r}|}{1-xz^{\lambda_r}}\right|^{\frac{1}{2}}\exp\left[\frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r}(|\,\psi(r\omega)\,| - \psi(r\xi))\right]\exp\Psi(\xi).$$

By the argument above we have

$$\prod_{r=1}^{\infty}\left|\frac{1-|\,xz^{\lambda_r}|}{1-xz^{\lambda_r}}\right| < e^{-AN(|\alpha|)}$$

provided that Δ is sufficiently small. Now $|\,\psi(r\omega)\,| \leqslant \psi(r\xi)$ for all ω, so that

$$|\,G(x,z)\,| \leqslant e^{-\frac{1}{2}AN(|\alpha|)}\exp\tfrac{1}{2}[\,|\,\psi(\omega)\,| - \psi(\xi)]\exp\Psi(\xi). \tag{38}$$

We shall assume that as $\xi \to 0$, $\xi\psi'''(\xi) = O(\psi''(\xi))$, which by using (32) implies (11). Now we have

$$\psi(\omega) = \psi(\xi) + i\eta\psi'(\xi) - \tfrac{1}{2}\eta^2\psi''(\xi) + O(\eta^3\psi'''(\xi)),$$

$$|\psi(\omega)|^2 = [\psi(\xi) - \tfrac{1}{2}\eta^2\psi''(\xi)]^2 + \eta^2[\psi'(\xi)]^2 + O(\eta^3\psi(\xi)\psi'''(\xi))$$

$$= [\psi(\xi)]^2 + \eta^2([\psi'(\xi)]^2 - \psi(\xi)\psi''(\xi)) + O(\eta^3\xi^{-1}\psi(\xi)\psi''(\xi)).$$

Hence

$$|\psi(\omega)| - \psi(\xi) = \tfrac{1}{2}\eta^2\frac{[\psi'(\xi)]^2 - \psi(\xi)\psi''(\xi)}{\psi(\xi)} + O\left(\frac{\eta^3}{\xi}\psi''(\xi)\right).$$

We shall assume that

$$[\psi'(\xi)]^2 < \theta\psi(\xi)\psi''(\xi) \tag{39}$$

for sufficiently small ξ, where $\theta < 1$. Then it follows that

$$|\psi(\omega)| - \psi(\xi) \leqslant -A\eta^2\psi''(\xi)$$

if ω is in a sufficiently small Stolz angle Δ. This implies, using (33) and (38), that

$$|J_{C_2}| \leqslant \frac{1}{4\pi^2}\iint_{C_1}\left|\frac{G(x,z)}{x^{m+1}z^{n+1}}dx\,dz\right| = o(\xi\{\Psi''(\xi)\}^{-\frac{1}{2}}\exp[\Psi(\xi)+n\xi]). \tag{40}$$

Writing

$$F(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K(\alpha)\,e^{\alpha y}\,d\alpha \tag{41}$$

we deduce finally that

$$p_m(n) = \frac{\xi}{\sqrt{(2\pi)}}\{\Psi''(\xi)\}^{-\frac{1}{2}}F((m-m_0)\,\xi)\exp[\Psi(\xi)+n\xi] + o(\xi\{\Psi''(\xi)\}^{-\frac{1}{2}}\exp[\Psi(\xi)+n\xi]), \tag{42}$$

provided only that the conditions stated above are satisfied. Under these conditions it also follows that

$$p(n) \sim \frac{1}{\sqrt{(2\pi)}}\{\Psi''(\xi)\}^{-\frac{1}{2}}\exp[\Psi(\xi)+n\xi]. \tag{43}$$

Further, it follows similarly, by considering the integral

$$\frac{1}{(2\pi i)^2}\iint\frac{G(x,z)}{x^{m+1}z^{n+1}}(1-x)^r(1-z)^s\,dx\,dz,$$

that

$$\Delta_m^r\Delta_n^s p_m(n) = \frac{\xi^{r+s+1}}{\sqrt{(2\pi)}}F^{(r)}((m-m_0)\,\xi)\{\Psi''(\xi)\}^{-\frac{1}{2}}\exp[\Psi(\xi)+n\xi] + o(\xi^{r+s+1}\{\Psi''(\xi)\}^{-\frac{1}{2}}\exp[\Psi(\xi)+n\xi]), \tag{44}$$

where Δ_m and Δ_n denote difference operators with respect to the variables m and n (e.g. $\Delta_n p_m(n) = p_m(n) - p_m(n-1)$).

The above results may be stated in the form of the

THEOREM. *Let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ be a sequence of positive integers such that*

(i) $\Sigma\lambda_r^{-2}$ *converges.*

Then let $\psi(\omega)$ be the function of the complex variable $\omega = \xi + i\eta$ defined in the region $\xi > 0$ by

$$\psi(\omega) = \sum_{r=1}^{\infty} e^{-\lambda_r\omega},$$

and suppose that

(ii) $\sum_{t=1}^{\infty} t^2\psi'''(t\xi) = O(\psi'''(\xi))$ *as* $\xi \to 0$,

(iii) $\{\psi'(\xi)\}^2 < \theta\psi(\xi)\psi''(\xi)$ *for some fixed* $\theta < 1$ *and for* ξ *sufficiently small,*

(iv) $\xi\psi'''(\xi) = O(\psi''(\xi))$ *as* $\xi \to 0$,

(v) *for sufficiently small* ξ, $|\psi(\omega)| < \theta\psi(\xi)$ *in the region* $\xi\Delta < |\eta| \leqslant \pi$ *for any fixed* Δ *and some* $\theta < 1$ *depending only on* Δ.

Then writing
$$\Psi(\omega) = \sum_{r=1}^{\infty} \frac{1}{r}\psi(r\omega),$$
$$m_0 = m_0(\xi) = \sum_{r=1}^{\infty} \frac{1}{e^{\lambda_r \xi}-1},$$
$$K(\alpha) = \prod_{r=1}^{\infty}\left(1+\frac{\alpha}{\lambda_r}\right)^{-1} e^{\alpha/\lambda_r}$$
and
$$F(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K(\alpha)\, e^{\alpha y}\, d\alpha, \tag{41}$$
it follows that as $n \to \infty$ the number of partitions $p_m(n)$ of n into m parts λ_r satisfies
$$\Delta_m^r \Delta_n^s p_m(n) = \xi^{r+s+1} F^{(r)}((m-m_0)\,\xi)\, p(n) + o(\xi^{r+s+1} p(n)), \tag{45}$$
where ξ is the root of the equation $\Psi'(\xi)+n = 0$ and Δ_m and Δ_n denote difference operators with respect to the variables m and n, so that $\Delta_n p_m(n) = p_m(n) - p_m(n-1)$, etc.

Also the total number $p(n)$ of partitions of n into parts λ_r satisfies
$$p(n) \sim \frac{1}{\sqrt{(2\pi)}}\{\Psi''(\xi)\}^{-\frac{1}{2}} \exp[\Psi(\xi)+n\xi]. \tag{43}$$
In particular
$$p_m(n) = \xi F((m-m_0)\,\xi)\, p(n) + o(\xi p(n)). \tag{42}$$

$F(y)$ is a distribution function in the usual statistical sense. It is indefinitely differentiable and satisfies
$$\int_{-\infty}^{\infty} F(y)\, dy = 1.$$

It follows from our results that it is non-negative, although we have not constructed a direct proof of this result. As we have seen the moment-generating function of $F(y)$ is
$$K(\alpha) = \prod_{r=1}^{\infty}\left(1+\frac{\alpha}{\lambda_r}\right)^{-1} e^{\alpha/\lambda_r}.$$

This implies that the mean of the distribution occurs at $y = 0$ or $m = m_0$, and that the second semi-invariant with respect to the y variable is $\Sigma\lambda_r^{-2}$, which we have assumed to be finite.

In most of the cases that we shall consider $F(y)$ will have a maximum at y_1, where the second derivative $F''(y_1)$ is different from zero and will be such that $F(y_1) > F(y)$ for all $y \neq y_1$. Since $F''(y)$ is continuous, and since $F''(y_1) \neq 0$, we must have
$$F''(y) < -c, \quad |y-y_1| < 2h$$
for some strictly positive constants c and h. It then follows from our result (45) that the difference $\Delta_m^2 p_m(n)$ is negative for sufficiently large n, in the corresponding range of m. Also $F'(y) > ch$ for $y_1 - 2h < y < y_1 - h$ and $F'(y) < -ch$ for $y_1 + h < y < y_1 + 2h$, and it follows from (45) that, for sufficiently large n, $\Delta_m p_m(n)$ is positive and negative respectively in the corresponding ranges of m. But since $\Delta_m^2 p_m(n)$ is negative, $\Delta_m p_m(n)$ is monotonic and strictly decreasing for values of m corresponding to $|y-y_1| < 2h$. Hence there exists m_1 in this range such that for $m < m_1$, $\Delta_m p_m(n)$ is positive and for $m > m_1$, $\Delta_m p_m(n)$ is negative, for m in the range. Now for y outside the range $|y-y_1| < h$, $F(y) < F(y_1) - \delta$ for some positive δ, since $F(y)$ is continuous and $F(y) \to 0$ as $y \to \pm\infty$. Thus, using (42), $p_m(n) < p_{m_1}(n)$ for values of m corresponding to values of y outside the range $|y-y_1| < h$.

Thus there are at most two consecutive values of m for which $p_m(n)$ attains its maximum value for a fixed but sufficiently large value of n. We shall later use this to prove the conjecture of Auluck, Chowla and Gupta (1), in the case of partitions into the natural numbers.

It will be seen that the value of m at which the maximum occurs will be asymptotic to the value of m at which the mean of the distribution occurs, unless $\Sigma\lambda_r^{-1}$ converges.

It is often useful to know something of the asymptotic behaviour of $F(y)$ as $y \to \pm\infty$. To determine this behaviour as $y \to +\infty$ we proceed as follows:

$$F(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K(\alpha)\, e^{\alpha y}\, d\alpha.$$

$K(\alpha)$ has poles at the points $\alpha = -\lambda_r$. Shifting the contour across these poles we obtain the expansion

$$F(y) \sim \sum_r C_r\, e^{-\lambda_r y}, \tag{46}$$

which is asymptotic in the sense of Poincaré. The C_r are constants for those λ_r which are not repeated, but, if λ_r is repeated ν_r times, C_r is a polynomial of degree $\nu_r - 1$ in y.

It is more difficult to obtain general results for the behaviour of $F(y)$ as y becomes negative. If the series $\Sigma\lambda_r^{-1}$ is convergent we observe that

$$K(\alpha) = O(|\alpha|^{-2}\exp[\Re\alpha\Sigma\lambda_r^{-1}]),$$

as $\Re\alpha \to \infty$ uniformly in $\Im\alpha$. Hence if $y \leqslant -\Sigma\lambda_r^{-1}$

$$F(y) = O\left(\frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} |\alpha|^{-2}\,|d\alpha|\right).$$

Making $c \to +\infty$ we deduce

$$F(y) = 0 \quad \text{for} \quad y \leqslant -\Sigma\lambda_r^{-1}. \tag{47}$$

This result may be explained by the fact that

$$m_0(\xi) = \Sigma\frac{1}{e^{\lambda_r\xi}-1} < \xi^{-1}\Sigma\lambda_r^{-1}$$

when the series is convergent. Thus

$$y = \{m - m_0(\xi)\}\,\xi \leqslant -\Sigma\lambda_r^{-1}$$

only for negative m, in which case the number of partitions is naturally zero.

It will be shown later, by studying various special cases, that the distribution function $F(y)$ is highly asymmetric if λ_r increases rapidly with r, but if λ_r does not increase very rapidly with r the distribution becomes nearly normal.

If the series $\Sigma\lambda_r^{-1}$ is divergent the following rough argument shows that we may use the saddle-point method for both variables of integration, provided only that certain conditions, of the same nature as we have imposed above, are satisfied. For we have

$$\frac{\partial^2}{\partial\alpha^2}\log g(\alpha,\omega) = \sum_1^\infty \frac{\omega^2\, e^{(\lambda_r+\alpha)\omega}}{(e^{(\lambda_r+\alpha)\omega}-1)^2}.$$

Now as $\omega \to 0$ for $\alpha \neq -\lambda_r$,

$$\frac{\omega^2\, e^{(\lambda_r+\alpha)\omega}}{(e^{(\lambda_r+\alpha)\omega}-1)^2} \to \frac{1}{(\lambda_r+\alpha)^2}.$$

Thus if $\Sigma\lambda_r^{-2}$ diverges

$$\frac{\partial^2}{\partial\alpha^2}\log g(\alpha,\omega) \to \infty.$$

This is a necessary condition for it to be possible to expand the integrand in (16) as a Taylor series over a sufficient length of the path of integration for us to be able to apply the saddle-point method to the x variable (see p. 226 above). It is still necessary to impose some conditions on the λ_n, but we shall not study this problem in detail as we must use methods slightly different from those of this paper.

It then follows, under the assumption of suitable conditions, that the orthodox results of statistical mechanics are correct. It also follows that the distribution function $F(y)$ represents a normal distribution (i.e. it is the function $\frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}v^2}$ when an appropriate scale factor has been introduced and a change of origin has been made.)

4. *A special class of cases.* In this section we shall make the assumption that $N(\lambda) \sim A\lambda^\beta$ where A is a positive constant, $0 < \beta < 2$ and $\beta \neq 1$. This will enable us to obtain asymptotic formulae for some of the quantities involved in the above analysis and to deduce that some of our conditions are satisfied. We notice that $\Sigma\lambda_r^{-2}$ is convergent since $\beta < 2$.

It is first necessary to obtain an asymptotic formula for $\psi(\xi)$ which may be done by a generalization of the well-known theorem of Abel on power series:

$$\psi(\xi) \sim A\Gamma(\beta+1)\,\xi^{-\beta}. \tag{48}$$

Similarly,

$$\psi'(\xi) \sim -A\beta\Gamma(\beta+1)\,\xi^{-\beta-1}, \tag{49}$$

$$\psi''(\xi) \sim A\beta(\beta+1)\,\Gamma(\beta+1)\,\xi^{-\beta-2}, \tag{50}$$

etc. Hence the conditions (ii), (iii) and (iv) are satisfied and

$$-n = \Psi'(\xi) = \sum_{t=1}^{\infty} \psi'(t\xi)$$

$$\sim -A\beta\Gamma(\beta+1)\,\zeta(\beta+1)\,\xi^{-\beta-1},$$

$$\xi \sim \{A\beta\Gamma(\beta+1)\,\zeta(\beta+1)\}^{1/(\beta+1)}\,n^{-1/(\beta+1)}, \tag{51}$$

$$m_0(\xi) \sim \xi^{-1}\Sigma\lambda_r^{-1} \quad \text{for} \quad \beta < 1, \tag{52}$$

and

$$m_0(\xi) = \sum_{t=1}^{\infty} \psi(t\xi) \sim A\Gamma(\beta+1)\,\zeta(\beta)\,\xi^{-\beta} \quad \text{for} \quad \beta > 1. \tag{53}$$

Also

$$\Psi(\xi) = \sum_{1}^{\infty} \frac{1}{r}\psi(r\xi)$$

$$\sim A\Gamma(\beta+1)\,\zeta(\beta+1)\,\xi^{-\beta}.$$

Thus

$$\Psi(\xi) + n\xi \sim A(\beta+1)\,\Gamma(\beta+1)\,\zeta(\beta+1)\,\xi^{-\beta}. \tag{54}$$

Then if the condition (v) is satisfied, we can conclude that

$$\log p(n) \sim \{A\Gamma(\beta+1)\,\zeta(\beta+1)\}^{1/(\beta+1)}\,(\beta+1)\left(\frac{n}{\beta}\right)^{\beta/(\beta+1)}. \tag{55}$$

We can obtain some additional information about the behaviour of $F(y)$ as y decreases. We have

$$F(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K(\alpha)\,e^{\alpha y}\,d\alpha.$$

In the case $\beta < 1$ we make a change of origin

$$F(y-\Sigma\lambda_r^{-1}) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} K_1(\alpha)\, e^{\alpha y}\, d\alpha,$$

where
$$K_1(\alpha) = \prod_{r=1}^{\infty}\left(1+\frac{\alpha}{\lambda_r}\right)^{-1}.$$

By a theorem on integral functions with real negative zeros (see Titchmarsh (9), p. 271) we deduce

$$\log K_1(\alpha) \sim -\pi A \alpha^{\beta} \operatorname{cosec} \pi\beta \tag{56}$$

as $|\alpha| \to \infty$ with $|\arg\alpha| < \pi - \delta$. We may use this result to prove the asymptotic formula for

$$\log F(y-\Sigma\lambda_r^{-1}) \sim (\beta^{1/(1-\beta)} - \beta^{\beta/(1-\beta)})\,(\pi A \operatorname{cosec} \pi\beta)^{1/(1-\beta)}\, y^{-\beta/(1-\beta)}, \tag{57}$$

as $y \to +0$, by using the saddle-point method.

We may extend the range of validity of this formula to the case $\beta > 1$ (but not the case $\beta = 1$) if we allow for the change of sign of some of the terms. Then

$$\log F(y) \sim (\beta^{1/(1-\beta)} - \beta^{\beta/(1-\beta)})\,(-\pi A \operatorname{cosec} \pi\beta)^{1/(1-\beta)}\,(-y)^{\beta/(\beta-1)}, \tag{58}$$

as $y \to -\infty$. In order to prove this result it is necessary to use the result analogous to (56) for integral functions of order $\beta > 1$, which may be proved similarly.

5. *Particular types of partition.* We shall now consider the application of the above theory to particular types of partition function.

(*a*) *The case $\lambda_r = r$.* This case has been dealt with by Erdös and Lehner (4) and later by Auluck, Chowla and Gupta (1). Their results are easily verified. It has also been studied by Szekeres (7). It is trivial that our conditions are satisfied.

We have

$$K(\alpha) = \prod_{r=1}^{\infty}\left(1+\frac{\alpha}{r}\right)^{-1} e^{\alpha/r}$$
$$= e^{C\alpha}\,\Gamma(\alpha+1),$$

where C is Euler's constant.

We may easily obtain approximate formulae for $\psi(\xi)$, ξ, $m_0(\xi)$ and then for $F((m-m_0)\,\xi)$. This gives the results that have previously been obtained by Auluck, Chowla and Gupta (1). Also

$$F(y) = \exp\{-(C+y) - e^{-(C+y)}\}.$$

Thus the distribution function of $p_m(n)$ has a unique maximum, and so, by our remarks on the partial differences of $p_m(n)$, we deduce that $p_m(n)$ attains its maximum value for at most two consecutive values of m for sufficiently large fixed n. This proves the conjecture of Auluck, Chowla and Gupta (1).

(*b*) *The case $\lambda_r = r^{\kappa}$, where κ is an integer and $\kappa > 1$.* Here the conditions (i), (ii), (iii) and (iv) are satisfied trivially. The condition (v) is also satisfied; the proof of this fact is rather difficult, but it follows from various results on the function $\psi(\omega)$ which is used in work on Waring's problem (see, for example, Landau (6), equations (273) and (274)). Since $\Sigma\lambda_r^{-1}$ is convergent, we see that the distribution is very asymmetric, having mean and standard deviation of the same order. The distribution has a finite cut-off as shown by (47). We see that $N(\lambda) \sim \lambda^{1/\kappa}$ and obtain the results

$$\Psi(\xi) \sim \Gamma\left(1+\frac{1}{\kappa}\right)\zeta\left(1+\frac{1}{\kappa}\right)\xi^{-1/\kappa},$$

$$\Psi'(\xi) \sim -\frac{1}{\kappa}\Gamma\left(1+\frac{1}{\kappa}\right)\zeta\left(1+\frac{1}{\kappa}\right)\xi^{-1/\kappa-1},$$

$$\xi \sim \left\{\frac{1}{\kappa}\Gamma\left(1+\frac{1}{\kappa}\right)\zeta\left(1+\frac{1}{\kappa}\right)\right\}^{\kappa/(\kappa+1)} n^{-\kappa/(\kappa+1)},$$

$$m_0(\xi) \sim \xi^{-1}\zeta(\kappa)$$

and
$$\log p(n) \sim \left\{\Gamma\left(1+\frac{1}{\kappa}\right)\zeta\left(1+\frac{1}{\kappa}\right)\right\}^{\kappa/(\kappa+1)}\left(1+\frac{1}{\kappa}\right)\kappa^{1/(\kappa+1)}n^{1/(\kappa+1)}.$$

Thus the value of m for which the maximum of $p_m(n)$ occurs is of order $A(\kappa)\,n^{\kappa/(\kappa+1)}$.

We again suspect that there is a unique maximum, but in order to prove this we should have to examine the distribution function in detail. We shall not carry out the analysis here for general κ, but we shall obtain a simpler form of $F(y)$ in the case of partitions into squares.

We can determine asymptotic formulae for $p(n)$ from Ingham's Theorem 2, but these results have previously been obtained by Wright (10). In the case of partitions into squares the function $K(\alpha)$ is an elementary function. We have

$$K(\alpha) = e^{-\frac{1}{6}\pi^2\alpha}\prod_{r=1}^{\infty}\left(1+\frac{\alpha}{r^2}\right)^{-1} = e^{-\frac{1}{6}\pi^2\alpha}\frac{\pi i\alpha^{\frac{1}{2}}}{\sin(\pi i\alpha^{\frac{1}{2}})}.$$

From this we deduce that for $y > 0$

$$F\left(y-\frac{\pi^2}{6}\right) = -2\sum_{r=1}^{\infty}(-1)^r r^2 e^{-r^2y} = \frac{d}{dy}\vartheta_3\left(\frac{1}{2}\middle|\frac{iy}{\pi}\right),$$

in the notation of Tannery and Molk. It is easily proved by direct computation that the maximum of

$$-2\sum_{r=1}^{\infty}(-1)^r r^2 e^{-r^2y}$$

occurs at $y = 0{\cdot}9054\ldots$. Since the mean of the distribution occurs at $y = \frac{1}{6}\pi^2 = 1{\cdot}6449\ldots$ it follows that the mean and the maximum are different in this case. Asymptotic formulae for $p_m(n)$ follow from those for $p(n)$ if we use (42).

(c) *Prime numbers.* Here we take λ_r to be the rth prime which, by the prime number theorem, is asymptotic to $r\log r$. This case is of some interest, in that it demonstrates the power of the above methods; however, it is necessary to use some additional arguments to prove the results. The distribution function shows some remarkable properties; this is due to the fact that the series $\Sigma 1/p$ is 'only just' divergent.

The conditions (i), (ii), (iii) and (iv) are satisfied trivially. The condition (v) is not satisfied, but in view of some work of Vinogradoff and Linnik on Goldbach's conjecture, an account of which is given by Tchudakoff (8), we are able to state some results which are sufficient for our purpose. The difficulty arises from the fact that there is only one even prime number (namely 2) and therefore the generating function $F(x,z)$ has a singularity at the point $x = -1$, $z = -1$ which gives rise to a contribution to the contour integral which is almost of the same order of magnitude as the contribution from the singularity at the point $x = 1, z = 1$.

If none of the numbers λ_r are even we see that the function $F(x,z)$ satisfies the functional equation

$$F(-x, -z) = F(x, z),$$

so that the singularities at $x = -1$, $z = -1$ and at $x = 1$, $z = 1$ have equal weight. This reflects the fact that in this case it is not possible to partition an even number into an odd number of odd parts. In the case of prime numbers, however, the function $F(x,z)$ satisfies the functional equation

$$F(-x, -z) = \frac{1-xz^2}{1+xz^2} F(x,z),$$

which shows that the contribution to the integral from the singularity at $(-1, -1)$ is of a strictly smaller order of magnitude than that from the singularity at $(1, 1)$. This means that we can prove the asymptotic formula for $p_m(n)$ if we use the results of Vinogradoff and Linnik. These state that if we exclude a region in a Stoltz angle about the point $z = 1$ $(\omega = i\pi)$ the condition (v) is then satisfied.

We may not, however, obtain a similar result for the differences of $p_m(n)$. In the case of the first differences (with respect to either variable) the contributions from the singularities are of the same order of magnitude. Thus we cannot expect to prove any analogue of the Auluck, Chowla and Gupta conjecture. Computation leads us to believe that such an analogue is not true.

In the case of partitions into prime numbers we have

$$\sum_{\lambda_r < X} \lambda_r^{-1} \sim \log\log X.$$

Hence

$$m_0(\xi) = \sum_r \frac{1}{e^{\lambda_r \xi} - 1} \sim \frac{1}{\xi} \log\log\frac{1}{\xi}.$$

Also

$$\psi(\xi) \sim \frac{1}{\xi \log 1/\xi},$$

$$\Psi(\xi) \sim \frac{\pi^2}{6\xi \log 1/\xi},$$

$$\Psi'(\xi) \sim -\frac{\pi^2}{6\xi^2 \log 1/\xi},$$

$$\xi \sim \frac{\pi}{\sqrt{3}} \frac{1}{(n \log n)^{\frac{1}{2}}}$$

and

$$\log p(n) \sim \frac{2\pi}{\sqrt{3}} \left(\frac{n}{\log n}\right)^{\frac{1}{2}}.$$

The above asymptotic formula for $\log p(n)$ is well known. If we refer back to the formulae of our theorem we obtain an asymptotic formula for $p(n)$ itself. This formula, however, involves transcendental sums over the primes. It is possible to express these transcendental sums in terms of the zeros of the Riemann zeta function. We may also state asymptotic formulae for $p_m(n)$ in this case. It should be noticed that the above results have been obtained without any recourse to the Riemann hypothesis on the zeros of the zeta function, though it may be necessary to use the Riemann hypothesis to express the sums explicitly in terms of the zeros of $\zeta(s)$. Brigham (3) has obtained an asymptotic formula for a certain weighted partition function corresponding to partitions into prime powers, in terms of the zeros of the zeta function, on the assumption of the Riemann hypothesis.

It may be verified that $F(y)$ lies between the orders

$$e^{-e^{e^{-(1\pm\epsilon)y}}}$$

as $y \to -\infty$ for any fixed positive ϵ. Also $F(y) \sim A\,e^{-2y}$ as $y \to +\infty$. The maximum of $p_m(n)$ occurs for

$$m \sim \frac{\sqrt{3}}{\pi}(n \log n)^{\frac{1}{2}} \log\log n.$$

The mean of the distribution is of the same order as the value corresponding to the maximum, but may differ from this value by

$$O((n \log n)^{\frac{1}{2}}).$$

Thus in this case the ratio $m_{\text{mean}}/m_{\text{max.}}$ only tends to unity very slowly as $n \to \infty$. We have asymptotic equality because $\Sigma 1/p$ diverges.

We may apply the above methods to partitions of other similar types. For example, our conditions are generally satisfied in the following cases:

(*d*) Numbers λ_r which are representable as the sum of s κth powers provided that $s < 2\kappa$.

(*e*) Numbers λ_r of the form $[Ar^{\sigma}(\log r)^{\tau}]$, provided that $\sigma > \frac{1}{2}$.

In particular we may deal with partitions into numbers represented as the sum of two or three squares, counted according to the number of representations, which have applications in physics to the two- and three-dimensional Bose-Einstein gases.

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