Percolation on interacting, antagonistic networks

Kun Zhao[1] and Ginestra Bianconi[1]

[1]*Department of Physics, Northeastern University, Boston, Massachusetts 02115 USA*

Recently, new results on percolation of interdependent networks have shown that the percolation transition can be first order. In this paper we show that, when considering antagonistic interactions between interacting networks, the percolation process might present a bistability of the equilibrium solution. To this end, we introduce antagonistic interactions for which the functionality, or activity, of a node in a network is incompatible with the functionality, of the linked nodes in the other interacting networks. In particular, we study the percolation transition in two interacting networks with purely antagonistic interaction and different topology. For two antagonistic Poisson networks of different average degree we found a large region in the phase diagram in which there is a bistability of the steady state solutions of the percolation process, i.e. we can find that either one of the two networks might percolate. For two antagonistic scale-free networks we found that there is a region in the phase diagram in which, despite the antagonistic interactions, both networks are percolating. Finally we characterize the rich phase diagram of the percolation problems on two antagonistic networks, the first one of the two being a Poisson network and the second one being a scale-free network.

PACS numbers: 64.60.ah, 05.70.Jk, 89.75.Hc

arXiv:1210.7498v1 [physics.soc-ph] 28 Oct 2012

Over the last ten years percolation processes, and more in general, dynamical processes in complex networks [1, 2], have gathered great attention. In this context it has been shown that complex topologies strongly affect the dynamics occurring in networks. However, many complex systems involve interdependencies between different networks, and accounting for these interactions is crucial in economic markets, interrelated technological and infrastructure systems, social networks, diseases dynamics, and human physiology. Recently, important new advances have been made in the characterization of percolation [3–10] and other dynamical processes [11–16] on interacting and interdependent networks. In these systems, one network function depends on the operational level of other networks. A failure in one network could trigger failure avalanches in the other interdependent network, resulting in the increased fragility of the interdependent system. In fact, it has been shown [3–7] that two interdependent networks are more fragile than a single network and that the percolation transitions in interdependent networks can be first order. These results have subsequently been extended to multiple interdependent networks [8, 9] and to networks in which only a fraction of the nodes are interdependent [10].

Here, we want to investigate the role of antagonistic interactions in the percolation transition between interacting networks. As it is happening in spin systems, where antiferromagnetic interactions can result in the frustration of the system, also in interacting network, the presence of antagonistic interactions between the nodes introduce further complexity in the percolation problem. As a first step in investigating this complexity in this paper we will consider two interacting networks with purely antagonistic interactions. We will show that for two Poisson networks with exclusively antagonistic interactions the steady state of the percolation dynamics corresponds to the percolation of one of the two networks. Nevertheless, the solution of the model is surprising because there is a wide region of the phase space in which there is a bistability of the percolation process: depending on the detail of the percolation dynamics either one of the two networks might end up to be percolating. Therefore, in this new percolation problem, not only the percolation transitions might be first order, but we found that there is a real hysteresis in the system as we modify the average degrees of the two networks. Furthermore, we extend the analysis to networks with other topologies, studying the percolation transition in two antagonistic scale-free networks, and in two networks one of which is a Poisson network, and the other one is a scale-free network. We characterize the rich phase digram of the percolation transition in these networks that display both first and second order transitions, and, in top of that, might show a bistability of the percolation solutions. Interestingly, in the percolation phase diagram of these interacting networks there is a region in which both networks percolate on the same time, demonstrating a strong interplay between the the percolation dynamics and the topology of the network. Finally, these results shed new light on the complexity that the percolation process acquires, when considering percolation on interdependent, antagonistic networks.

Percolation on antagonistic networks. In this paper we introduce antagonistic interactions in percolation dynamics on interdependent networks.The difference with respect to the case of interdependent networks is that if a node i is active on one network it cannot be active in the other one. We consider two networks of N nodes. We call the networks, network A and network B with degree distribution $p^A(k), p^B(k)$ respectively. Each node i is represented in both networks. In particular, each node has a set of neighbor nodes j in network A, i.e. $j \in N^A(i)$ and a set of neighbor nodes j in network B, i.e. $j \in N^B(i)$.

A node i belongs to the percolation cluster of network A, if it has at least one neighbor $j \in N^A(i)$ in the percolating cluster of network A, and has no neighbors $j \in N^B(i)$ in network B that belong to the percolating cluster of network B. Similarly, A node i belongs to the percolation cluster of network B, if it has at least one neighbor $j \in N^B(i)$ in the percolating cluster of network B, and has no neighbors $j \in N^B(i)$ in network A that belong to the percolating cluster of network A.

If we define S_A as the probability to find a node in the percolation cluster of network A, and S_B as the probability to find a node in the percolation cluster of network B, we have

$$\begin{aligned} S_A &= [1 - G_0^A(1 - S'_A)]G_0^B(1 - S'_B) \\ S_B &= [1 - G_0^B(1 - S'_B)]G_0^A(1 - S'_A), \end{aligned} \tag{1}$$

where S'_A is the probability that, following a link in network A, we find a node in the percolation cluster of network A, and S'_B is the probability that, following a link in network B, we reach a node in the percolation cluster of network B. Moreover, in Eq. (1) we have used $G_0^{A/B}(z)$ and $G_1^{A/B}(z)$ to indicate the generating functions of network A and B defined according to the definition

$$\begin{aligned} G_1(z) &= \sum_k \frac{kp_k}{\langle k \rangle} z^{k-1} \\ G_0(z) &= \sum_k p_k z^k, \end{aligned} \tag{2}$$

where we use the degree distributions $p^A(k), p^B(k)$, respectively, for network A and network B. According to percolation theory, applied in this case to two antagonistic networks, the variables S'_A, S'_B satisfy, on a locally tree like network, the following recursive equations

$$\begin{aligned} S'_A &= (1 - G_1^A(1 - S'_A))G_0^B(1 - S'_B) = f_A(S'_A, S'_B), \\ S'_B &= (1 - G_1^B(1 - S'_B))G_0^A(1 - S'_A) = f_B(S'_A, S'_B). \end{aligned} \tag{3}$$

The solutions to the recursive Eqs. (3) can be classified into three categories:

- *(i) The trivial solution in which neither of the network is percolating $S'_A = S'_B = 0$.*
- *(ii) The solutions in which just one network is percolating.* In this case we have either $S'_A > 0, S'_B = 0$ or $S'_B > 0, S'_A = 0$. From Eqs. (3) we find that the solution $S'_A > 0, S'_B = 0$ emerges at a critical line of second order phase transition, characterized by the condition

$$\left. \frac{dG_1^A(z)}{dz} \right|_{z=1} \equiv \frac{\langle k(k-1) \rangle_A}{\langle k \rangle_A} = 1. \tag{4}$$

  Similarly the solution $S'_B > 0, S'_A = 0$ emerges at a second order phase transition when we have $\frac{\langle k(k-1) \rangle_B}{\langle k \rangle_B} = 1$. This condition is equivalent to the critical condition for percolation in single networks, as it should, because one of the two networks is not percolating.
- *(iii) The solutions for which both networks are percolating.* In this case we have $S'_A > 0, S'_B > 0$. This solution can either emerge (a) at a critical line indicating a continuous phase transition or (b) at a critical line indicating discontinuous phase transition. For situation (a) the critical line can be determined by imposing, for example, $S'_A \to 0$ in Eqs. (3), which yields

$$\begin{aligned} S'_B &= 1 - G_1^B(1 - S'_B), \\ 1 &= \frac{\langle k(k-1) \rangle_A}{\langle k \rangle_A} G_0^B(1 - S'_B). \end{aligned} \tag{5}$$

  A similar system of equation can be found by using Eqs. (3) and imposing $S'_B \to 0$. For situation (b) the critical line can be determined imposing that the curves $S'_A = f_A(S'_A, S'_B)$ and $S'_B = f_B(S'_A, S'_B)$, are tangent to each other at the point where they intercept. This condition can be written as

$$\left(\frac{\partial f_A}{\partial S'_A} - 1 \right) \left(\frac{\partial f_B}{\partial S'_B} - 1 \right) - \frac{\partial f_A}{\partial S'_B} \frac{\partial f_B}{\partial S'_A} = 0, \tag{6}$$

  where S'_A, S'_B must satisfy the Eqs. (3).

The stability of the solutions. Not every solution of the recursive Eqs. (3) is stable. Therefore, we check the stability of the fixed points solutions of Eqs. (3) by linearizing the equations around each solution. The Jacobian matrix J of the system of Eqs. (3) is given by

$$J = \begin{vmatrix} \frac{\partial f_A}{\partial S'_A} & \frac{\partial f_A}{\partial S'_B} \\ \frac{\partial f_B}{\partial S'_A} & \frac{\partial f_B}{\partial S'_B} \end{vmatrix}. \tag{7}$$

The eigenvalues $\lambda_{1,2}$ of the Jacobian can be found by solving the characteristic equation $|J - \lambda I| = 0$, which reads for our specific problem,

$$\left(\frac{\partial f_A}{\partial S'_A} - \lambda \right) \left(\frac{\partial f_B}{\partial S'_B} - \lambda \right) - \frac{\partial f_A}{\partial S'_B} \frac{\partial f_B}{\partial S'_A} = 0. \tag{8}$$

A solution is stable if and only if $\lambda_{1,2} < 1$. Assuming that the eigenvalues of the Jacobian corresponding to each solution of the Eqs. (3) change continuously when we smoothly change the parameters determining the topology of the networks, the change of stability of each solution will occur when $\max(\lambda_1, \lambda_2) = 1$. In the following we will discuss the stability of the solutions of type (i)-(iii).

- *(i) Stability of the trivial solution $S'_A = S'_B = 0$.* The solution is stable as long as the following two conditions are satisfied:

$$\lambda_{1,2} = \frac{\langle k(k-1) \rangle_{A/B}}{\langle k \rangle_{A/B}} < 1. \tag{9}$$

  Therefore the stability of this solution change on the critical lines $\frac{\langle k(k-1) \rangle_A}{\langle k \rangle_A} = 1$ and $\frac{\langle k(k-1) \rangle_B}{\langle k \rangle_B} = 1$.

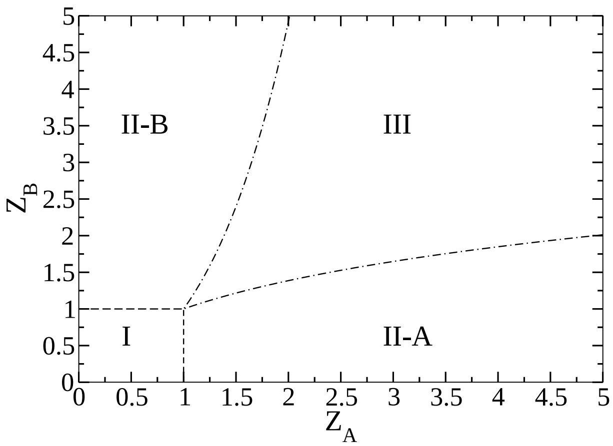

FIG. 1: Phase diagram of the percolation process on two antagonistic Poisson networks of average degree $\langle k\rangle_A = z_A$ and $\langle k\rangle_B = z_B$ respectively.

- *(ii) Stability of the solutions in which only one network is percolating.* For the case of $S'_A = 0$ $S'_B > 0$ the stability condition reads

$$\lambda_1 = \left.\frac{G_1^B(z)}{dz}\right|_{z=1-S'_B} < 1$$
$$\lambda_2 = \frac{\langle k(k-1)\rangle_A}{\langle k\rangle_A} G_0^B(1-S'_B) < 1. \qquad (10)$$

  We note here that if $\lambda_2 > \lambda_1$ we expect to observe a change in the stability of the solution on the critical line given by Eqs. (5). A similar condition holds for the stability of the solution $S'_A > 0, S'_B = 0$.

- *(iii) Stability of the solution in which both networks are percolating* $S'_A > 0, S'_B > 0$ For characterizing the stability of the solutions of type (iii) we have to solve Eq. (8) and impose that the eigenvalues $\lambda_{1,2}$ are less then 1, i.e. $\lambda_{1,2} < 1$. We observe here that for $\lambda = 1$ Eq. (8) reduces to Eq. (6). Therefore we expect to have a stability change of these solutions on the critical line given by Eq. (6).

Two Poisson networks. In order to consider a specific example of antagonistic networks we consider two antagonistic Poisson networks with average degree $\langle k\rangle_A = z_A$ and $\langle k\rangle_B = z_B$ respectively. In the following we characterize the phase diagram of the percolation dynamics on two antagonistic Poisson networks shown in Fig. 1.

In region I, with $z_A < 1, z_B < 1$, the only stable solution is the trivial solution $S'_A = S'_B = 0$.

In regions II-A, with $z_A > 1$ and $1 < z_B < \ln(z_A)/(1-1/z_A)$, the only stable solution is $S'_A > 0, S'_B = 0$. Network A has a larger average connectivity than network B and is percolating, while network B is non percolating.

In regions II-B, with $z_B > 1$ and $1 < z_A < \ln(z_B)/(1-1/z_B)$, the only stable solution is $S'_B > 0, S'_A = 0$. Network B has a larger average connectivity than network A and is percolating, while network A is non percolating.

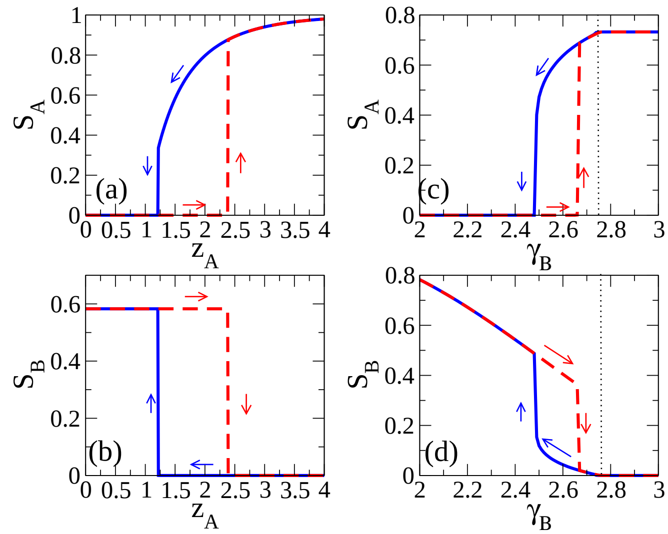

FIG. 2: (Color online) Panels (a) and (b) show the hysteresis loop for the percolation problem on two antagonistic Poisson networks with $z_B = 1.5$. Panels (c) and (d) show the hysteresis loop for the percolation problem on two antagonistic networks of different topology: a Poisson network of average degree $z_A = 1.8$ and a scale-free networks with power-law exponent γ_B, minimal degree $m = 1$ and maximal degree $K = 100$. The hysteresis loop is performed using the method explained in the main text. The value of the parameter ϵ used in this figure is $\epsilon = 10^{-3}$.

In region III, with $z_A > \ln(z_B)/(1-1/z_B)$ and $z_B > \ln(z_A)/(1-1/z_A)$,there are two stable solutions $S'_A > 0, S'_B = 0$ and $S'_B > 0, S'_A = 0$. In this region, both network A and network B might be percolating. The system is bistable and Eqs. (3) has two possible solutions.

One should note that the solution $S'_A > 0, S'_B > 0$ in which both networks are percolating is always unstable in this case. This implies that at any given time only one of the two networks is percolating.

In order to demonstrate the bistability of the percolation solution in region III we solved recursively the Eqs. (3) for $z_B = 1.5$ and variable values of z_A (see Figure 2). We start from values of $z_A = 4$, and we solve recursively the Eqs. (3). We find the solutions $S'_A = S'_A(z_A = 4) > 0$, $S'_B = S'_B(z_A = 4) = 0$.Then we lower slightly z_A and we solve again the Eqs. (3) recursively, starting from the initial condition $S'^o_A = S'_A(z_A = 4)+\epsilon$, $S'^o_B = S'_B(z_A = 4)+\epsilon$, and plot the result. (The small perturbation $\epsilon > 0$ is necessary in order not to end up with the trivial solution $S'_A = 0, S'_B = 0$.) Using this procedure we show that if we first lower the value of z_A and then again we raise it, spanning the region III of the phase diagram as shown in Figure 2, the solution present an hysteresis loop. This means that in the region III either network A or network B might end up to be percolating depending on the details of the percolation dynamics.

Two scale-free networks. Here, we characterize the phase digram of two antagonistic scale-free networks with power-law exponents γ_A, γ_B, as shown in Figure 3. The

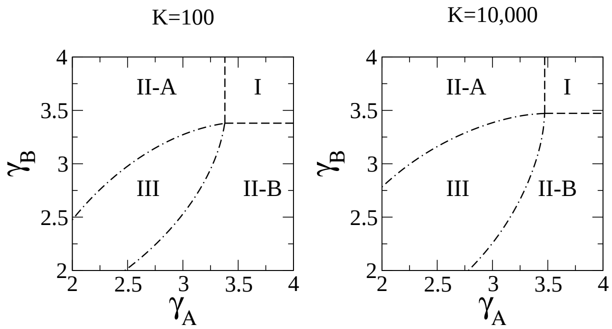

FIG. 3: The phase diagram of the percolation process in two antagonistic scale-free networks with power-law exponents γ_A, γ_B for finite networks. The minimal degree of the two networks is $m = 1$ and the maximal degree K. The panel on the left show the effective phase diagram with $K = 100$ and the panel on the right show the effective phase diagram for $K = 10,000$.

two networks have minimal connectivity $m = 1$ and varying value of the maximal degree K.

The critical lines of the phase diagram depend on the value of the maximal degree K of the networks. Therefore, the plot in Figure 3 has to be considered as the effective phase diagrams of the percolation problem on antagonistic networks with a finite cutoff K. The phase diagram is rich, showing a region (Region III) in the figure where both networks are percolating demonstrating an interesting interplay between the percolation dynamics and the topology of the network.

A description of the phase diagram follows.

In region I, with $\frac{\langle k(k-1)\rangle_A}{\langle k\rangle_A} < 1$ and $\frac{\langle k(k-1)\rangle_B}{\langle k\rangle_B} < 1$, the trivial solution $S'_A = 0, S'_B = 0$ is stable.

In regions II-A the solution in which network A percolates $S'_A > 0, S'_B = 0$ is stable.

In regions II-B the solution in which network B percolates $S'_B > 0, S'_A = 0$ is stable.

In region III the solution in which both network A and B percolate $S'_A > 0, S'_B > 0$ is stable.

A Poisson network and a scale-free network. Finally we consider the case of a Poisson network (network A) with average connectivity $\langle k\rangle_A = z_A$, and a network B with scale-free degree distribution and power-law exponent of the degree distribution γ_B. The scale-free network has minimal connectivity $m = 1$ and maximal degree given by K. In Figure 4 we show the phase diagram of the model in the plane (γ_B, z_A). The critical lines of the phase diagram are dependent on the value of the cutoff K of the scale-degree distribution. For these reasons we have to consider the phase diagrams in Figure 4 as effective phase diagrams of the percolation problem on networks with maximal degree K. The phase diagram includes two regions, (region III and region V) with bistability of the solutions and two regions (region IV and region V) in which the solution in which both networks are percolating is stable. In the following we describe the percolation stable solution in the different regions of the phase diagram in Figure 4.

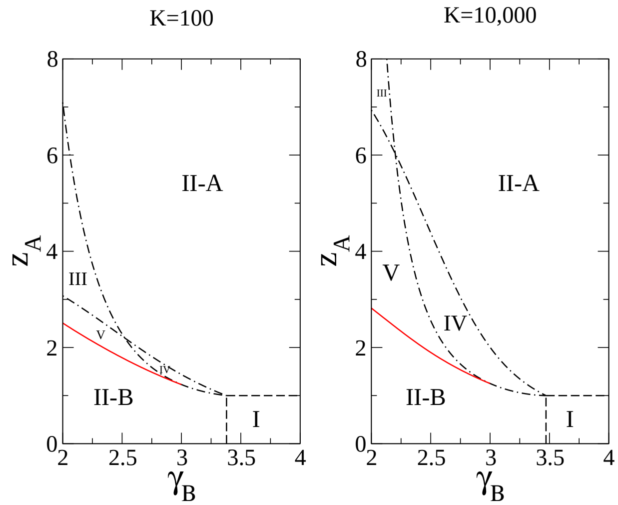

FIG. 4: (Color online) Phase diagram of the percolation process on a Poisson network with average degree $\langle k\rangle_A = z_A$ interacting with a scale-free network of power-law exponent γ_B, minimal degree $m = 1$ and maximal degree K. The panel on the left show the effective phase diagram for $K = 100$ and the panel on the right show the effective phase diagram for $K = 10,000$.

In region I, with $z_A < 1$, $\frac{\langle k(k-1)\rangle_B}{\langle k\rangle_B} < 1$, the only stable solution is the trivial solution $S'_A = S'_B = 0$.

In region II-A the solution for which network A is percolating $S'_A > 0, S'_B = 0$ is stable.

In regions II-B the solution for which network B is percolating $S'_B > 0, S'_A = 0$ is stable.

In region III there are two stable solutions, in which either network A percolates $S'_A > 0, S'_B = 0$ or network B percolates $S'_B > 0, S'_A = 0$. Therefore in this region we have bistability of the solutions.

In region IV there is only one stable solution in which both networks percolate, $S'_A > 0, S'_B > 0$.

In region V there are two stable solutions. In the first one, network B percolates $S'_B > 0, S'_A = 0$, and in the second one, both networks percolate $S'_A > 0, S'_B > 0$. In this region there is a bistability of the solutions of the percolation problem.

In order to demonstrate the bistability of the percolation problem we solved recursively the Eqs. (3) for $z_B = 1.8$ (see Figure 2). We start from values of $\gamma_B = 3$, and we solve the Eqs. (3). We find the solutions $S'_A = S'_A(\gamma_B = 3) > 0$, $S'_B = S'_B(\gamma_B = 3) = 0$. Then we lower slightly γ_B and we solve again the Eqs. (3) recursively, starting from the initial condition $S'^o_A = S'_A(\gamma_B = 3) + \epsilon$, $S'^o_B = S'_B(\gamma_B = 3) + \epsilon$ and we plot the result. (The small perturbation $\epsilon > 0$ is necessary in order not to end up with the trivial solution $S'_A = 0, S'_B = 0$.) Using this procedure we show that if we first lower the value of γ_B and then again we raise it as shown in Figure 2, the solution present first a second order phase transition to a phase in which both networks are percolating and then an hysteresis loop in correspondence of region V. This demonstrates the bistability of the solutions in region V and the existence of a phase in which both network percolate in region IV and region V.

Conclusions. In conclusions, we have investigated how much antagonistic interactions modify the phase diagram of the percolation transition. The percolation process on two antagonistic networks shows important new physics of the percolation problem. In fact, the percolation process in this case shows a bistability of the solutions. This implies that depending on the details of the percolation dynamics, the steady state of the system might change. In particular, we have demonstrated the bistability of the percolation solution for the percolation problem on two antagonistic Poisson networks, or two antagonistic networks with different topology: a Poisson network and a scale-free network. Moreover, in the percolation transition between two scale-free antagonistic networks and in the percolation transition between two antagonistic networks with a Poisson network and a scale-free networks, we found a region in the phase diagram in which both networks are percolating, despite the presence of antagonistic interactions. We believe that this paper opens new perspectives in the percolation problem on interdependent networks, which might include both interdependencies and antagonistic interactions eventually combined in a boolean rule. In an increasingly interconnected world, understanding how much these different types of interactions affect percolation transition is becoming key to answering fundamental questions about the robustness of interdependent networks.

[1] S. N. Dorogovtsev, A. Goltsev and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008);

[2] A. Barrat, M. Barthélemy, A. Vespignani *Dynamical Processes on complex Networks* (Cambridge University Press, Cambridge, 2008).

[3] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley and S. Havlin, Nature **464** 1025 (2010).

[4] A. Vespignani, Nature **464**, 984 (2010).

[5] S.-W. Son, G. Bizhani, C. Christensen, P. Grassberger and M .Paczuski, EPL **97** 16006 (2012).

[6] G. J. Baxter, S. N. Dorogovtsev, A. V. Goltsev and J. F. F. Mendes, arXiv:1207.0448v1.

[7] R. Parshani, S. V. Buldyrev, and S.Havlin, Physical Review Letters **105** 048701 (2010).

[8] J. Gao, S.V. Buldyrev, H.E. Stanley and S. Havlin, Nature Physics **8**, 40 (2012).

[9] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, Physical Review Letters **107** 195701 (2011).

[10] R. Parshani, S. V. Buldyrev and S. Havlin, PNAS, **108** 1007 (2011).

[11] S. Gomez, A. Diaz-Guilera, J. Gomez-Gardeñes, C. J. Perez-Vicente, Y. Moreno, A. Arenas, arXiv:1207.2788

[12] E. Cozzo, A. Arenas, Y. Moreno, arXiv:1205.3111

[13] J. Gómez-Gardeñes, I. Reinares, A. Arenas and L.M. Floría Scientific Reports **2** 620 (2012)

[14] A. Bashan, R. P. Bartsch, J. W. Kantelhardt, S. Havlin, P. Ch. Ivanov Nature Communications **3** 702 (2012).

[15] Kyu-Min Lee, K. -I. Goh and I. -M. Kim Jour.Kor. Phys. Soc. **60**, 641 (2012).

[16] K.-M. Lee, J. Y. Kim, W.-k. Cho, K.-I. Goh and I.-M. Kim, New J. Phys. **14**, 033027 (2012).