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Asymptotic Relations for Partitions

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Asymptotic expansions, similar to those of Roth and Szekeres, are obtained for the number of partitions of a positive integer into summands from a given set of integers under new restrictions.

1. INTRODUCTION

Let $A = \{a_0, a_1, ...\}$ be an infinite sequence of monotonically increasing positive integers. Under suitable restrictions on A, we shall obtain asymptotic estimates for the number $p_A(n)$ of ways that n can be written in the form

$$n = a_{\nu_1} + a_{\nu_2} + \cdots + a_{\nu_v} \quad (0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \cdots \leqslant \nu_v),$$

v being arbitrary. We shall assume from now on that the greatest common divisor of A is one.

To state our result requires some definitions and notation that will be used throughout this paper:

Let $A(u)$ denote the number of elements of A which are $\leqslant u$.

We define the function f_A for real $x > 0$ by

$$f_A(x) = \sum e^{-xa_\nu}.$$

We say that A has property (I) if with $\epsilon > 0$ an arbitrary constant, μ any fixed positive integer,

$$\sum (xa_\nu)^\mu e^{-xa_\nu} = O(f_A^{1+\epsilon}(x))$$

and

$$f_A(x)/f_A[x(1 - f_A^{-(1+\epsilon)/3}(x))] = O(1)$$

as $x \to 0$. It shall be proven later (see Lemmas 2.4 and 2.5) that A has property (I) when any of the conditions (i), (ii), (iii) below hold:

$$\text{(i)} \qquad s = \lim_{\nu\to\infty} \frac{\log\log a_\nu}{\log \nu} \text{ exists;}$$

$$\text{(ii)} \qquad \varliminf_{\nu\to\infty}(\log a_\nu)/\nu > 0;$$

$$\text{(iii)} \qquad A(2u) = O\{A(u)\} \quad \text{as} \quad u \to \infty.$$

We say that A has property (II) if there exists some constant δ with $1 > \delta > 0$ and some constant η with $1/3 > \eta > 0$ such that

$$A(x^{-1} f_A^{-\delta}(x))/\log f_A(x) \to \infty$$

and

$$A(x^{-1}) > f_A^{(2/3)+\eta}(x)$$

as $x \to 0$. It shall be proven later that A has property (II) when either

$$(\varlimsup_{\nu\to\infty} \log a_\nu/\log \nu)/\varliminf_{\nu\to\infty} \log a_\nu/\log \nu < \tfrac{3}{2}$$

or $A(2u) = O\{A(u)\}$ as $u \to \infty$ or (i) above holds with $s > 0$.

We say that A is a P-sequence if there does not exist a number p such that $p | a_\nu$ for all sufficiently large a_ν. Note that A is a P-sequence iff it has the property P_k of Bateman and Erdös [1] for all k.

We define α, A_μ, D_ρ to be the η^*, A_μ^*, and D_ρ^* of Roth and Szekeres [2], that is, $\alpha = \alpha(n)$ is determined from

$$n = \sum a_\nu (e^{\alpha a_\nu} - 1)^{-1}. \tag{1.1}$$

$A_\mu = A_\mu(n)$ $(\mu = 2, 3,...)$ is defined by

$$A_\mu = \sum a_\nu^\mu g_\mu(e^{\alpha a_\nu})(e^{\alpha a_\nu} - 1)^{-\mu},$$

where $g_\mu(x)$ is a certain polynomial of degree $\leqslant \mu - 1$ and in particular $g_1(x) = 1$ and $g_2(x) = x$.

$D_\rho = D_\rho(n)$ $(\rho = 1, 2,...)$ is defined by

$$D_\rho = A_2^{-6\rho} \sum_{\mu_1=2}^{\infty} \cdots \sum_{\mu_{5\rho}=2}^{\infty} d_{\mu_1\mu_2\ldots\mu_{5\rho}} A_{\mu_1} A_{\mu_2} \cdots A_{\mu_{5\rho}}$$

the summation being subject to

$$\mu_1 + \mu_2 + \cdots + \mu_{5\rho} = 12\rho,$$

where the d's are certain numerical constants.

Our main result can now be stated.

THEOREM 1.1. *Let A have properties* (I) *and* (II). *Suppose that either A is a P-sequence or that*

$$\lim_{x\to 0} \log f_A(\alpha)/\log \alpha = 0.$$

Suppose furthermore that

$$\overline{\lim} \frac{\log\log a_\nu}{\log \nu} < \infty.$$

Let m be any fixed integer $\geqslant 2$. Then

$$p_A(n) = (2\pi A_2)^{-1/2} \exp\left\{\sum_{\nu=0}^{\infty} \left[\frac{\alpha a_\nu}{e^{\alpha a_\nu} - 1} - \log(1 - e^{-\alpha a_\nu})\right]\right\}$$
$$\times \left[1 + \sum_{\rho=1}^{m-2} D_\rho + O\{f_A^{1-(2m/3)}(\alpha)\}\right].$$

It is clear even from the statement of the theorem that we rely heavily upon the work of Roth and Szekeres [2]. An advantage of this theorem over that of Roth and Szekeres is that the condition $\log a_\nu = O(\log \nu)$ is relaxed. Another is that the theory of trigonometric sums is not required. (See the remark at the end of this paper.)

The condition $A(2u) = O\{A(u)\}$ as $u \to \infty$ has been used previously by Schwarz, for example, [3–5]. Theorem 1.1 is most closely related to Theorems 5 and 6 of [5]. We only consider the case of integral a_ν's; however, in this case, our restrictions complement those of Theorem 5 and are rather more easily verified than those of Theorem 6 of [5].

Proof of Theorem 1.1. The proof is rather long. Thus, we sketch very briefly the proof before proceeding. It is well known that the generating function $F_A(z)$ for $p_A(n)$ (with $p_A(0) = 1$) is

$$F_A(z) = \sum p_A(n)\, z^n = \prod (1 - z^{a_\nu})^{-1},$$

where the series and product converge absolutely for $|z| < 1$ (see [6]). From Cauchy's theorem,

$$p_A(n) = \frac{1}{2\pi i} \int F_A(\xi)\, \xi^{-(n+1)}\, d\xi, \tag{1.2}$$

where the path of integration is taken to be a circle with center at the origin and radius $\rho < 1$. For reasons explained in detail in [7] we set $\rho = e^{-\alpha}$, where α is defined by 1.1. (This is a saddle-point condition.)

Letting $\xi = e^{-\alpha+i\theta}$ in (1.2), we obtain

$$p_A(n) = \frac{1}{2\pi} \exp\left\{\alpha n - \sum \log(1 - e^{-\alpha a_\nu})\right\}$$
$$\times \int_{-\pi}^{\pi} \exp\left\{-\sum \log\left(\frac{1 - \exp(-\alpha a_\nu + i a_\nu \theta)}{1 - \exp(-\alpha a_\nu)}\right) - in\theta\right\} d\theta. \quad (1.3)$$

We dissect the integration of (1.3) into 3 parts:

$$\int_{-\pi}^{\pi} = I_1 + I_2 + I_3, \qquad I_1 = \int_{-\theta_0}^{\theta_0}, \qquad I_2 = \int_{\theta_0}^{\pi}, \qquad I_3 = \int_{-\pi}^{-\theta_0}.$$

If we set

$$\theta_0 = \alpha f_A^{-(1+\eta)/3}(\alpha), \quad (1.4)$$

we may approximate the integrand of I_1 by its Taylor series about $\theta = 0$ and the contributions from I_2 and I_3 are negligible. In all subsequent discussion, we assume that θ_0 has this value.

Moreover, in the following discussions, all equations involving α may be satisfied only for sufficiently small α. In view of (1.1), this is equivalent to sufficiently large n.

2. The Asymptotic Expansion of I_1

We now determine the asymptotic expansion of I_1 with θ_0 given by Eq. (1.4). Though Section 2 draws very heavily upon [2], there are differences since we are not assuming that $\lim \log a_\nu / \log \nu$ exists as $\nu \to \infty$. We attempt to indicate these while avoiding as much duplication of [2] as possible. The Taylor series of the integrand about $\theta = 0$ is required. Let

$$f_\nu(\theta) = -\log\left(\frac{1 - \exp(i a_\nu \theta - \alpha a_\nu)}{1 - \exp(-\alpha a_\nu)}\right).$$

In a manner very similar to that in [2], it can be shown that for $\theta \leqslant \alpha/2$, we may write

$$f_\nu(\theta) = \sum_{\mu=1}^{\infty} \frac{b_\mu}{\mu!} (i\theta a_\nu)^\mu, \quad (2.1)$$

where

$$b_\mu = \underbrace{\sum_{l=1}^{\mu} \sum_{k_1=1} \cdots \sum_{k_l=1}}_{k_1+k_2+\cdots+k_l=\mu} (e^{\alpha a_\nu} - 1)^{-l} \frac{\mu!}{k_1! k_2! \ldots k_l!}. \quad (2.2)$$

Thus

$$b_\mu = (e^{\alpha a_\nu} - 1)^{-\mu} g_\mu(e^{\alpha a_\nu}),$$

where $g_\mu(x)$ is that of Section 1. Proceeding as in [2], we obtain (fixed integral $m \geqslant 2$)

$$\begin{aligned}\sum_{\nu=0}^{\infty} f_\nu(\theta) &= \sum_{\nu=0}^{\infty} \left(\sum_{\mu=1}^{2m-1} \frac{b_\mu}{\mu!} (i\theta a_\nu)^\mu \right) \\ &+ O \left\{ \sum_{\nu=0}^{\infty} \left[\frac{(|\theta| a_\nu)^{2m}}{(e^{\beta a_\nu} - 1)^{2m-1}} + \frac{(|\theta| a_\nu)^{2m}}{(e^{\beta a_\nu} - 1)} + \frac{(|\theta| a_\nu)^{2m}}{(e^{\beta a_\nu} - 1)^{2m}} \right] \right\} \qquad (2.3)\end{aligned}$$

where $\beta = \alpha(1 - f_A^{-(1+\eta)/3}(\alpha))$. The 0-term depends upon m but not of course α.

The following lemma is easily established:

LEMMA 2.1. *Let μ and j be integers with $\mu \geqslant j$. Then*

$$\sum_{\nu=0}^{\infty} (\alpha a_\nu)^\mu (e^{\alpha a_\nu} - 1)^{-j} = O \left\{ \sum_{\nu=0}^{\infty} [1 + (\alpha a_\nu)^\mu] e^{-\alpha a_\nu} \right\},$$

where the O-constant depends upon μ and j but is independent of α.

From Lemma 2.1 and Eq. (2.3), Lemma 2.2 easily follows.

LEMMA 2.2. *Let A have property* (I). *Then with $\epsilon > 0$ an arbitrary constant*

$$\sum_{\nu=0}^{\infty} f_\nu(\theta) = \sum_{\nu=0}^{\infty} \left(\sum_{\mu=1}^{2m-1} \frac{b_\mu}{\mu!} (i\theta a_\nu)^\mu \right) + O\{(\theta/\alpha)^{2m} f_A^{1+\epsilon}(\alpha)\}.$$

We now obtain as in [2] that

$$I_1 = \int_{-\theta_0}^{\theta_0} \exp \left\{ -\tfrac{1}{2} A_2 \theta^2 + \sum_{\mu=3}^{2m-1} \frac{A_\mu}{\mu!} (i\theta)^\mu + O\{(\theta/\alpha)^{2m} f_A^{1+\epsilon}(\alpha)\} \, d\theta \right\}. \qquad (2.4)$$

Note that in view of Eqs. (1.1), (2.2), and (2.3), the coefficient of θ is 0, which is of course the saddle-point condition at work.

LEMMA 2.3. *Let A have property* (I). *Let $\epsilon > 0$ be an arbitrary constant. Let $\mu \geqslant 2$ be a fixed integer. Then*

$$\alpha^{\mu} A_{\mu} = O\{f_A^{1+\epsilon}(\alpha)\}.$$

The O-constant depends upon μ but not α.

Proof. From the definition of the A_μ, it easily follows that

$$\alpha^{\mu} A_{\mu} = O\left\{\sum [1 + (\alpha a_\nu)^{\mu}]\, e^{-\alpha a_\nu}\right\},$$

as $\alpha \to 0$. Since A has property (I), the lemma follows.
From Lemma 2.3, we obtain that

$$A_{3+j}\theta^{3+j} = O\{f_A^{1+\epsilon-[(1+\eta)/3](3+j)}(\alpha)\}.$$

Since we are free to choose $\epsilon < 2m\eta/3$, we may expand the integrand in Eq. (2.2) as in [2] to obtain

$$\exp\{-(-\tfrac{1}{2}A_2\theta)^2\}\{1 + E(\theta) + O(f_A^{1-(2m/3)}(\alpha))\},$$

where

$$E(\theta) = \sum_{\nu=1}^{5m} \sum_{\mu_1=3}^{2m-1} \cdots \sum_{\mu_\nu=3}^{2m-1} \frac{1}{\mu!} \frac{A_{\mu_1} A_{\mu_2} \cdots A_{\mu_\nu}}{\mu_1! \mu_2! \ldots \mu_\nu!} \times (i\theta)^{\mu_1+\mu_2+\ldots\mu_\nu}.$$

Putting $\theta = t(\frac{1}{2}A_2)^{1/2}$, $B_0 = \theta_0(A_2/2)^{1/2}$, we obtain

$$I_1 = (\tfrac{1}{2}A_2)^{-1/2} \int_{-B_0}^{B_0} e^{-t^2}[1 + E\{t(\tfrac{1}{2}A_2)^{-1/2}\} + O(f_A^{1-(2m/3)}(\alpha))]\, dt.$$

It is easily seen that $\alpha^2 A_2 > K f_A(\alpha)$ for some constant $K > 0$ if A has property (I); hence, that $B_0 > K f_A^{\eta/3}(\alpha)$. We may thus replace the limits of integration by $-\infty$ to ∞. Expanding $E\{t(\frac{1}{2}A_2)^{-1/2}\}$ in powers of t and integrating termwise as in [2] yields the following.

THEOREM 2.1. *Let A have property* (I). *Let $m \geqslant 2$ be a fixed integer. Then*

$$I_1 = (2\pi/A_2)^{1/2}\left[1 + \sum_{\rho=1}^{m-2} D_\rho + O\{f_A^{1-(2m/3)}(\alpha)\}\right].$$

We now give some sequences which have property (I).

LEMMA 2.4. *A has property* (I) *if either*

$$s = \lim_{\nu\to\infty} \frac{\log\log a_\nu}{\log \nu}$$

exists or

$$\lim_{\nu\to\infty} \frac{\log a_\nu}{\nu} > 0.$$

Proof. (a) Suppose first of all that $s = 0$. We then have $a_\nu < \exp(v^\delta)$ for ν's greater than some lower bound $\nu_0 = \nu_0(\delta)$. Let a_{ν_1} be the largest element of A such that $\alpha a_\nu \leqslant 1$. Then

$$f_A(\alpha) > \sum_{\nu=\nu_0}^{\nu_1} e^{-\alpha a_\nu} > e^{-1}[A(1/\alpha) - \nu_0] > (e^{-1}/2)\, A(1/\alpha).$$

Since $A(1/\alpha) > 1/2 \log^{1/\delta} 1/\alpha$ and since $\delta > 0$ was an arbitrary constant, we obtain that $f_A(\alpha) > \log^{1/\psi} 1/\alpha$ for every constant $\psi > 0$.

(b) Let $\epsilon > 0$ be for now an arbitrary constant. Let αa_{ν_2} be the largest element of A such that $\alpha a_\nu \leqslant f_A^{\epsilon/\mu}(\alpha)$. Then

$$\sum_{\nu=\nu_2+1}^{\infty} (\alpha a_\nu)^\mu\, e^{-\alpha a_\nu} \leqslant \int_{a_{\nu_2}+1}^{\infty} (\alpha x)^\mu\, e^{-\alpha x}\, dx.$$

Thus

$$\sum_{\nu_2+1}^{\infty} = O\{\alpha^{-1} \exp(-f^{\epsilon/\mu}(\alpha)/4)\}.$$

From part (a), it follows that this O-term is $o(1)$. Since

$$\sum_{a_\nu \leqslant a_{\nu_2}} (\alpha a_\nu)^\mu\, e^{-\alpha a_\nu} \leqslant f_A^{1+\epsilon}(\alpha),$$

it follows that if $s = 0$, then A will have property (I) if $f_A(\beta) = O\{f_A^{1+\epsilon}(\alpha)\}$, where $\beta = \alpha(1 - f_A^{-(1+\epsilon)/3}(\alpha))$. Since $\beta \sim \alpha$ as $\alpha \to 0$, we obtain exactly as above that

$$\sum_{a_\nu > \beta^{-1} f_A^{\epsilon/\mu}(\alpha)} (\beta a_\nu)^\mu\, e^{-\beta a_\nu} = O(1).$$

For $\beta a_\nu \leqslant f_A^{\epsilon/\mu}(\alpha)$, $(\beta a_\nu)^\mu\, e^{-\beta a_\nu} = O\{(\alpha a_\nu)^\mu\, e^{-\alpha a_\nu}\}$, since we may choose ϵ small; thus, if $s = 0$, then A has property (I).

(c) Suppose now that $s > 0$. As in (*a*), we obtain, with $\delta > 0$ an arbitrary constant that

$$\log f_A(\alpha) < [1/(s + \delta)] \log\log 1/\alpha$$

as $\alpha \to 0$.

Now $y^{\mu}e^{-y}$ is monotonic decreasing for $y \geqslant c\ (= c(\mu))$. Let

$$f_{\mu}(\alpha) = \sum (\alpha a_{\nu})^{\mu} e^{-\alpha a_{\nu}} = \sum{}' + \sum{}'' ;$$

where $\sum'$ denotes summation over those ν for which $\alpha a_{\nu} \leqslant c$, and $\sum''$ denotes summation over the remaining ν. Clearly $\sum' = O(f_A(\alpha))$. By the same type of argument as in (b)

$$\sum{}'' < \int_{\alpha}^{\infty} y^{\mu-1}e^{-y} \frac{d}{dy} (\log^{1/(s-\delta)} y/\alpha)\, d\alpha$$

$$< \int_{e^2}^{\infty} y^{\mu-1}e^{-y} \log^{1/(s-\delta)} y/\alpha + O\{\log^{1/(s-\delta)} 1/\alpha\}.$$

Since $\log^{1/(s-\delta)} y/\alpha \leqslant \log^{1/(s-\delta)} y \log^{1/(s-\delta)} 1/\alpha$ for $y \geqslant e^2$ and sufficiently small α, we obtain that $\sum'' = O\{\log^{(s-\delta)^{-1}} 1/\alpha\}$. Hence, $\log f_{\mu}(\alpha) \sim s^{-1} \log\log 1/\alpha \sim \log f_A(\alpha)$.

Since $\beta \sim \alpha$, we may furthermore conclude that $\log f_A(\beta) \sim \log f_A(\alpha)$. Hence, A has property (I) when $s > 0$.

(d) Suppose $\underline{\lim}_{\nu\to\infty} \log a_{\nu}/\nu > 0$. Then as in (c), we obtain that $\sum'' = O(1)$; hence, $\log f_{\mu}(\alpha) = O\{f_A^{1+\epsilon}(\alpha)\}$. Furthermore, as in (b), it can be proven that $f_A(\beta) = O\{f_A^{1+\epsilon}(\alpha)\}$.

LEMMA 2.5. *If $A(2u) = O\{A(u)\}$ as $u \to \infty$, then there exists a positive constant c such that*

$$A(1/\alpha) > cf_A(\alpha).$$

Furthermore, A has property (I).

Proof. By assumption, there exists a positive constant c_0 such that $A(2u) < c_0 A(u)$ for $u > u_0$.

Hence

$$\sum_{a_{\nu} > 2^j\alpha^{-1}}^{a_{\nu} \leqslant 2^{j+1}\alpha^{-1}} e^{-\alpha a_{\nu}} < e^{-1}[A(2^{j+1}\alpha^{-1}) - A(2^j\alpha^{-1})] < e^{-2^j} c_0^{j+1} A(\alpha)^{-1}.$$

Thus $\sum \exp(-\alpha a_{\nu}) < c_0 A(\alpha^{-1}) \sum \exp(2^{-j})\, c_0{}^{j} = O\{A(\alpha^{-1})\}$ and the first part of the lemma is proven. Similarly, $\sum (\alpha a_{\nu})^{\mu} \exp(-\alpha a_{\nu}) = O\{A(\alpha^{-1})\}$. Furthermore, $f_A(\beta) = O\{A(\beta^{-1})\} = O\{A(\alpha^{-1})\}$ with $\beta = \alpha\{1 - f^{(-1+\epsilon)/3}(\alpha)\}$. Clearly $f_A(\alpha) > c_1 A(\alpha^{-1})$ and the lemma is proven.

It seems likely to the author that a fundamentally different method is required if A does not have property (I). (Probably ϵ being "small" in property (I) is sufficient.) There are sets which do not possess property (I). Define the sequence ν_i $(i = 0, 1, 2,\dots)$ with $\nu_0 = 0$ by

$$\nu_{i+1} = \left[\frac{3 + (4\exp(\nu_i^{1/2}) - 4\nu_i + 5)^{1/2}}{2}\right].$$

Define the sequence a_ν $(\nu = 0, 1, 2,\dots)$ by

$$a_{\nu_i} = [\exp(\nu_i^{1/2})];$$

$$a_{\nu_j+j} = a_{\nu_i} + j, \qquad i = 1, 2,\dots, \nu_{i+1} - \nu_i - 1.$$

Define α_i by

$$\exp(\nu_i^{1/2}) = \tfrac{1}{2}\alpha_i^{-1}(\log \alpha_i^{-1} + 3 \operatorname{loglog} \alpha_i^{-1}).$$

Then it can be shown by straightforward though somewhat messy calculations, again with $f_\mu(\alpha) = \sum (\alpha a_\nu)^\mu \exp(-\alpha a_\nu)$, that $f_\mu(\alpha_i) > f_A^{\mu/2}(\alpha_i)$.

Of course, one could have $a_{\nu_i-1} \approx \nu_i^s$ instead of ν_i^2 and $a_{\nu_i} \approx \exp(\nu_i^M)$ instead of $\exp(\nu_i^{1/2})$. The author knows of no sequence for which $\underline{\lim}\, \operatorname{loglog} a_\nu/\log \nu > 0$ that does not have property (I), however.

3. The Estimation of the Integrals I_2 and I_3

Since I_3 can be treated analogously to I_2, we shall consider only I_2. We set $I_2 = I_2' + I_2'' + I_2'''$, where the range of integration in I_2' is from θ_0 to α, in I_2'', from α to $\alpha f^\delta(\alpha)$, and in I_2''', from $\alpha f^\delta(\alpha)$ to π, where δ is fixed and $0 < \delta < 1/8$.

Throughout this section, we define $G(\theta)$ by

$$G(\theta) = \prod_{\nu=0}^{\infty} (1 - e^{-\alpha a_\nu})(1 - e^{-\alpha a_\nu + j a_\nu \theta})^{-1}.$$

Since $\operatorname{Re}(\log f(z)) = \frac{1}{2} \log |f(z)|^2$,

$$|G(\theta)| = \exp\left\{-\tfrac{1}{2} \sum \log\left(1 + \frac{2e^{\alpha a_\nu}(1 - \cos a_\nu \theta)}{(e^{\alpha a_\nu} - 1)^2}\right)\right\}.$$

Note that each term in the product is $\leqslant 1$ in modulus.

Lemma 3.1. *Suppose that for some constant η with η subject to $\frac{1}{3} \geqslant \eta > 0$*

$$A(x^{-1}) > f_A^{2/3+\eta}(x)$$

as $x \to 0$. Then for each constant $N > 0$

$$I_2' = O\{\alpha f_A^{-N}(\alpha)\}$$

as $\alpha \to 0$. The O-constant depends upon N and η but not α.

Proof. There exists a constant $K' > 0$ such that

$$1 - \cos a_\nu\theta > K'(a_\nu\theta)^2$$

for $a_\nu\theta < 1$. Also there exists a constant $K'' > 0$ such that

$$2e^{\alpha a_\nu}(e^{\alpha a_\nu} - 1)^{-2} > K''(\alpha a_\nu)^{-2}.$$

Thus $a_\nu\theta \leqslant a_\nu\alpha < 1$ implies that

$$|G(\theta)| < \exp\{-k/2 \log(1 + (K'/K'')(\theta^2/\alpha^2))\},$$

where k is the number of a_ν such that $\alpha a_\nu \leqslant 1$, i.e. $k = A(\alpha^{-1})$. This is

$$< \exp\{-k/2 \log(1 + K(\theta^2/\alpha^2))\}, \qquad K = K'/K''$$
$$\leqslant \exp\{-kKf_A^{-(2/3)-(2\eta/3)}(\alpha)\}.$$

Since $A(\alpha^{-1}) > f_A^{(2/3)+\eta}(\alpha)$

$$|G(\theta)| < \exp\{-f^{\eta/3}(\alpha)\} = O\{f_A^{-N}(\alpha)\}.$$

Moreover, in the same way, Lemma 3.2 follows.

LEMMA 3.2. *Suppose there exists a constant δ with $0 < \delta < 1$ such that*

$$A(f_A^{-\delta}(\alpha)\, \alpha^{-1})/\log f_A(\alpha) \to \infty$$

as $\alpha \to 0$. Let N be any constant >0. Then

$$I_2'' = O\{\alpha f^{-N}(\alpha)\}.$$

The O-constant depends upon δ and N but not α.

It seems that most "commonly occurring" sequences do satisfy property (II).

LEMMA 3.3. *A has property* (II) *if either*

(i) $s = \underline{\lim}_{\nu\to\infty} \dfrac{\log a_\nu}{\log \nu} \leqslant \overline{\lim}_{\nu\to\infty} \dfrac{\log a_\nu}{\log \nu} < 3s/2$;

(ii) $A(2u) = O\{A(u)\}$ *as* $u \to \infty$;

or

(iii) $l = \lim_{\nu\to\infty} \dfrac{\operatorname{loglog} a_\nu}{\log \nu}$ *exists and* $l > 0$.

Proof. (a) Suppose (i) holds. $s \geqslant 1$ since $a_\nu \geqslant \nu$. Reasoning as in (c) of Lemma 2.4, we conclude that $\log f_A(\alpha) < [(1+\epsilon)/s] \log(\alpha^{-1})$ as $\alpha \to 0$ for each constant $\epsilon > 0$.

Furthermore, it is easy to show that for each fixed η' with $0 \leqslant \eta' < 1$.

$$\log A(\alpha^{\eta'-1}) > \frac{2}{3} \frac{(1-\eta')(1-\epsilon)}{s} \log 1/\alpha \tag{3.1}$$

as $\alpha \to 0$. Thus there exists an $\eta > 0$ such that

$$A(\alpha^{-1}) > f_A^{(2/3)+\eta}(\alpha).$$

Finally, it is easy to show using (3.1) that there exists a δ with $1 > \delta > 0$ such that $A(\alpha^{-1} f_A^{-\delta}(\alpha))/\log f_A(\alpha) \to \infty$ as $\alpha \to 0$. Thus A has property (II). Case (iii) is similar.

(b) Suppose (ii) holds. Then there exists a positive constant $c > 0$ such that $A(2u) < cA(u)$ for $u > u_0$. This $\delta < (2 \log_2 c)^{-1}$. We may assume $f_A^{-\delta}(\alpha)\, \alpha^{-1} > u_0$. Then for $j = 1, 2,\ldots$

$$A(\alpha^{-1} 2^{-j}) > c^{-j} A(\alpha^{-1}).$$

Thus

$$\begin{aligned} A(\alpha^{-1} f_A^{-\delta}(\alpha)) &\geqslant A(\alpha^{-1} 2^{-(1+[\delta \log_2 f_A(\alpha)])}) \\ &\geqslant c^{-1} f_A^{-\delta \log_2 c}(\alpha) \\ &\geqslant c^{-1} f_A^{-1/2}(\alpha)\, A(\alpha^{-1}). \end{aligned}$$

From Lemma 2.5, we obtain

$$A(\alpha^{-1} f_A^{-\delta}(\alpha)) > c_1 f_A^{1/2}(\alpha).$$

This coupled with Lemma 2.5 shows that A has property (II).

The next lemma will be used repeatedly in the following.

LEMMA 3.4. *Let* ϕ *be any constant with* $0 < \phi < \pi$. *If* $\theta \in [\phi, 2\pi - \phi]$, *then*

$$|(1 - e^{-\alpha a_\nu})(1 - e^{-\alpha a_\nu + i a_\nu \theta})^{-1} \leqslant (1 - \cos \phi)^{-1/2} (1 - e^{-\alpha a_\nu})(1 + e^{-2\alpha a_\nu})^{-1/2}.$$

Proof. Suppose $\theta \in [\phi, \pi]$. If we construct the triangle with vertices at 0, 1, $\exp(-\alpha a_\nu + i a_\nu \theta)$, we obtain from the cosine law for triangles,

$$\begin{aligned} &|1 - \exp(-\alpha a_\nu + i a_\nu \theta)| \\ &\quad = (1 + \exp(-2\alpha a_\nu) - 2\exp(-\alpha a_\nu)\cos\theta)^{1/2} \\ &\quad = (1 + \exp(-2\alpha a_\nu))^{1/2}\left(1 - \frac{2\exp(-\alpha a_\nu)\cos\theta}{1 + \exp(-2\alpha a_\nu)}\right)^{1/2}. \end{aligned}$$

Since $0 < 2e^{-\alpha a_\nu}/(1 + e^{-2\alpha a_\nu}) \leqslant 1$, this yields that

$$\begin{aligned} |1 - \exp(-\alpha a_\nu + i a_\nu \theta)| &\geqslant (1 + \exp(-2\alpha a_\nu))^{1/2}(1 - \cos\theta)^{1/2} \\ &\geqslant (1 + \exp(-2\alpha a_\nu))^{1/2}(1 - \cos\phi)^{1/2}. \end{aligned}$$

The case $\theta \in [\pi, 2\pi - \phi]$ is treated similarly.

The estimation of I_2''' requires several theorems:

THEOREM 3.1. *Let A be a P-sequence. Let $\epsilon > 0$ be an arbitrary constant. Let $N > 0$ be an arbitrary fixed integer. Then*

$$\int_{\alpha^{\epsilon/N}}^{\pi} G(\theta)\, e^{-ni\theta}\, d\theta = O(\alpha^{N-\epsilon})$$

as $\alpha \to 0$. The O-constant depends upon N and ϵ but not α (or n).

Proof. For each $a_\nu \in A$, construct the sets

$$\Delta_{(\nu)}^{j} = \left[\frac{j2\pi - \alpha^{\epsilon/N}}{a_\nu}, \frac{j2\pi + \alpha^{\epsilon/N}}{a_\nu}\right],$$

where $1 \leqslant j \leqslant [a_\nu/2]$. We note that all $\theta \in \Delta_{(l)}^{j}$ are mapped into the interval $[j2\pi - \alpha^{\epsilon/N}, j2\pi + \alpha^{\epsilon/N}]$ under the map $\theta \to \theta a_\nu$.

Let $a_l > 1$ be some element of A and let (m, n) denote the greatest common divisor of the integers m and n. Suppose for some j the equation

$$\text{(i)} \qquad j/a_l = k/a_\nu$$

is solvable for each $a_\nu \in A$ with $\nu > l$. This implies that

$$[a_l/(a_l, j)]\, a_\nu$$

for each $\nu > l$. Since $j \leqslant a_l/2$, we have that $a_l/(a_l, j) \geqslant 2$ and this contradicts that A is a P-sequence. Thus, for each j there exists an $a(l, j) \in A$ such that (i) is not solvable for $a_\nu = a(l, j)$.

If $\theta \notin \bigcup \Delta_{(l)}^{j}$, $(j = 1, 2, \ldots, [a_l/2])$, then from Lemma 3.4

$$(1 - e^{-\alpha a_l})(1 - e^{-\alpha a_l + i a_l \theta})^{-1} = O(\alpha^{1-\epsilon/N})$$

as $\alpha \to 0$.

For sufficiently small α

$$\Delta^j_{(l)} \cap \bigcup_k \Delta^k_{(a(l,j))} = \Phi.$$

Thus if $\theta \in \Delta^j_{(l)}$ from Lemma 3.4, it follows that

$$[1 - \exp(-\alpha a(l,j))] \times [1 - \exp(-\alpha a(l,j) + ia(l,j)\theta)] = O(\alpha^{1-(\epsilon/N)}).$$

We thus conclude that the product

$$\frac{1 - e^{-\alpha a_l}}{1 - e^{-\alpha a_l + ia_l\theta}} \prod_j{}' \frac{1 - e^{-\alpha a(l,j)}}{1 - e^{-\alpha a_l + ia(l,j)}} = O(\alpha^{1-(\epsilon/N)}),$$

where $\prod_j'$ means that the product is taken over the distinct $a(l,j)$.

Let a_L be the smallest element of A which is greater than each of the $a(l,j)$. Then we may repeat the above argument with a_L replacing a_l. Doing this N times, we obtain that $G(\theta) = O(\alpha^{N-\epsilon})$ and the theorem follows at once.

THEOREM 3.2. *Let N and n be arbitrary fixed integers >0. Let*

$$\alpha^{1/4} = O\{f_A^{-N}(\alpha)\}$$

as $\alpha \to 0$ through elements of $\mathscr{A}$, where $\mathscr{A}$ is a subset of the positive reals and $\min \mathscr{A} = 0$. *Then*

$$\int_{\alpha^{1/4}}^{\pi} G(\theta)\, e^{-ni\theta}\, d\theta = O\{\alpha f_A^{-N}(\alpha)\}$$

as $\alpha \to 0$ through elements of $\mathscr{A}$. The O-constant depends upon N but not α (or n).

Proof. (a) We obtain as in the proof of Theorem 3.1 with $N = 1$, $\epsilon = 1/4$ that there is a subproduct of $G(\theta)$ which is $O(\alpha^{3/4})$.

(b) Now consider some higher indexed term:

$$(1 - \exp(-\alpha a_\nu))/(1 - \exp(-\alpha a_\nu + ia_\nu\theta)).$$

As in Theorem 3.1, construct the sets

$$\Delta^j_{(\nu)} = [(j2\pi - \alpha^{1/2})/a_\nu, \qquad (j2\pi + \alpha^{1/2})/a_\nu].$$

If $\theta \notin \bigcup \Delta^j_{(\nu)}$, then

$$(1 - \exp(-\alpha a_\nu))/(1 - \exp(-\alpha a_\nu + ia_\nu\theta) = O(\alpha^{1/2}).$$

Those $\theta \in \bigcup \Delta^{j}_{(\nu)}$ form a set of measure $\mathscr{M}$ where $\mathscr{M} = O\{\alpha^{1/2}\}$. Let

$$G_1(\theta) = \prod_{l \geqslant \nu} \frac{1 - e^{-\alpha a_l}}{1 - e^{-\alpha a_l + i a_l \theta}} .$$

Then

$$\begin{aligned} \int_{\alpha^{1/4}}^{\pi} | G(\theta)|| \, d\theta \,| &= O\left\{\alpha^{3/4} \int_{\alpha^{1/4}}^{\pi} | G_1(\theta)| \, d\theta\right\} \qquad \text{from part (a)} \\ &= O\left\{\alpha^{3/4} \int_{\theta \in \bigcup \Delta^{j}_{(\nu)}} | G_1(\theta)| \, d\theta + \alpha^{3/4} \int_{\theta \in \bigcup \Delta^{j}_{(\nu)}} | G_1(\theta)| \, d\theta\right\} \\ &= O\{\alpha^{1+1/4}\} = O\{\alpha f_A^{-N}(\alpha)\}. \end{aligned}$$

THEOREM 3.3. *Let N, n be arbitrary positive constants. Let δ be any constant with $0 < \delta < 1$. Define $M = [N/\delta] + 1$. Then*

$$\int_{\alpha f_A^{\delta}(\alpha)}^{3\pi/2a_M} G(\theta) \, e^{-in\theta} \, d\theta = O\{f_A^{-N}(\alpha)\}.$$

The O-constant depends upon N and δ but not α (or n).

Proof. Since $a_\nu \theta \leqslant 3\pi/2$ for $\nu = 1, 2,..., M$; $a_\nu \theta \in [\alpha f^{\delta}(\alpha), 3\pi/2]$ for these ν. From Lemma 3.4, it follows that if $\nu \leqslant M$

$$\frac{1 - e^{-\alpha a_\nu}}{1 - e^{-\alpha a_\nu + i a_\nu 2}} = O\{\alpha^{-1} f_A^{-\delta}(\alpha)\} \, 0(\alpha) = O\{f_A^{-\delta}(\alpha)\}.$$

Thus

$$| G(\theta)| = O\left\{\prod_{\nu \leqslant M} f_A^{-\delta}(\alpha)\right\} = O\{f_A^{-N}(\alpha)\}.$$

The theorem follows immediately.

THEOREM 3.4. *Let N and n be arbitrary constants >0. Let ψ be a constant >0 and suppose that $f_A(\alpha) = O(\alpha^{-\psi})$ as $\alpha \to 0$ through elements of $\mathscr{B}$, where* min $\mathscr{B} = 0$. *Let δ be any constant $0 < \delta < \frac{1}{2}$. Suppose furthermore that* $\overline{\lim}$ loglog $a_\nu/\log \nu < \infty$. *Then there exists a fixed integer M such that*

$$\int_{\alpha f_A^{\delta}(\alpha)}^{3\pi/2a_M} G(\theta) \, e^{-in\theta} \, d\theta = O\{\alpha f_A^{-N}(\alpha)\}$$

as $\alpha \to 0$ through elements of $\mathscr{B}$. The O-constant depends upon N and δ but not α or n.

Proof. Let $K = \overline{\lim}$ loglog $a_\nu/\log \nu$. Then $A(x) \geqslant (\log x)^{1/(K+\epsilon)}$ where $\epsilon > 0$ is an arbitrary constant.

(a) If $f_A{}^\delta(\alpha) \geqslant \alpha^{-1/(K+2)}$, let $M = [N/\delta] + [(2K+2)/\delta] + 2$. Since $a_\nu\theta \leqslant 3\pi/2$ for $\nu = 1, 2,..., M$, we obtain that

$$\int_{\alpha f_A{}^\delta(\alpha)}^{3\pi/2a_M} G(\theta)\, e^{-in\theta}\, d\theta = O\{f_A^{-\delta M}(\alpha)\}$$
$$= O\{\alpha f_A^{-N}(\alpha)\}$$

by the proof of Theorem 3.3.

(b) Here we suppose that $f_A{}^\delta(\alpha) < \alpha^{-1/(2K+2)}$. Suppose that $a_\nu \leqslant \alpha^{1-[1/(K+5/2)]}$. Then, by Lemma 3.4, as $\alpha \to 0$

$$(1 - \exp(-\alpha a_\nu))/(1 - \exp(-\alpha a_\nu + i a_\nu\theta)) \leqslant f_A{}^\delta(\alpha).$$

Now

$$A\{\alpha^{-(1-[1/(K+5/2)])}\} > \log^{1/(2K+1)}\{\alpha^{-(1-[1/(K+5/2)])}\}$$
$$\geqslant [1 - [1/(K+5/2)]]^{1/(2K+1)} \log^{1/(2K+1)}(1/\alpha).$$

Thus

$$A\{\alpha^{-(1-[1/(K+5/2)])}\} \geqslant c_0 \log^{1/(2K+1)}(1/\alpha),$$

where c_0 is some absolute constant >0. Now

$$\int_{\alpha f_A{}^\delta(\alpha)}^{\alpha(f_A(\alpha))^{\delta \log^{1/(4K+2)}(\alpha^{-1})}} |G(\theta)|\, d\theta \leqslant \alpha(f_A(\alpha))^{\delta \log^{1/(4K+2)}(\alpha^{-1})}$$
$$\times (f_A(\alpha))^{-\delta A(\alpha^{1-[1/(K+5/2)]})}$$
$$\leqslant \alpha(f_A(\alpha))^{-(\delta c_0/2)\log^{1/(2K+1)}(1/\alpha)}$$

as $\alpha \to 0$.

In the same way, we obtain

$$\int_{\alpha(f_A(\alpha))^{\delta \log^{j/(4K+2)}(1/\alpha)}}^{\alpha(f_A(\alpha))^{\delta \log^{(j+1)/(4K+2)}(1/\alpha)}} |G(\theta)|\, d\theta \leqslant \alpha(f_A(\alpha))^{-(\delta c_0/2)\log^{1/(2K+1)}(1/\alpha)},$$

as $\alpha \to 0$; for $j = 0, 1,..., -[-4K-2] = J$. Also

$$\int_{\alpha(f_A(\alpha))^{\delta \log^{J/(4K+2)}(\alpha^{-1})}}^{\alpha^{1-[1/(K+5/2)]}} |G(\theta)|\, d\theta$$
$$\leqslant \alpha^{1-[1/(K+5/2)]} \times (f_A(\alpha))^{-\delta \log^{J/(4K+2)}(\alpha)^{-1} \times A(\alpha^{-(1-(K+5/2))})}.$$
$$\leqslant \alpha^{1-[1/(K+5/2)]} (f_A(\alpha))^{-\delta c_0 \log^{(J+2)/(4K+2)}(\alpha^{-1})}.$$

Since $(J+1)/(4K+2) > 1$, we obtain that this last term is

$$O\{\alpha(f_A(\alpha))^{-\delta c_0 \log^{1/(4K+2)}(1/\alpha)}\}.$$

Since $\log^{1/(4K+2)}(\alpha^{-1}) > N/c_0\delta$ as $\alpha \to 0$, we obtain

$$\int_{\alpha f_A{}^{\delta}(\alpha)}^{\alpha^{1-[1/(K+5/2)]}} |\, G(\theta)|\, d\theta = O\{\alpha f_A^{-N}(\alpha)\}.$$

The integral from $\alpha^{1-[1/(K+5/2)]}$ to $3\pi/2a_M$ may be treated as in part (a); thus, the theorem is proven.

We now give our final result in the estimation of I_2.

THEOREM 3.5. *Let A have property* (II). *Choose N to be any fixed integer >0. Let A satisfy either* (i) *or* (ii) *below:*

(i) $\lim_{\alpha\to 0} \log f_A(\alpha)/\log 1/\alpha = 0$;

(ii) *A is a P-sequence.*

Suppose furthermore that $\overline{\lim} \log\log a_\nu/\log \nu < \infty$. *Then* $I_2 = O\{\alpha f_A^{-N}(\alpha)\}$.

Proof. Let δ be any possible δ of property (II) which is less than 1/8. We conclude from Lemmas 3.1 and 3.2 that

$$\int_{\theta_0}^{\alpha f_A{}^{\delta}(\alpha)} G(\theta)\, e^{-in\theta}\, d\theta = O\{\alpha f_A^{-N}(\alpha)\}. \tag{3.2}$$

(a) Suppose condition (i) holds. Then we take $\psi = N^{-1}$ and $\mathscr{B}$ to be the positive real in Theorem 3.4. Then it follows that

$$\int_{\alpha f_A{}^{\delta}(\alpha)}^{3\pi/2a_{M_1}} G(\theta)\, e^{-in\theta}\, d\theta = O\{\alpha f_A^{-N}(\alpha)\}. \tag{3.3}$$

Moreover, in case (i) the set $\mathscr{A}$ in Theorem 3.2 may be chosen to be the positive reals. For sufficiently small α, $\alpha^{1/4} < 3\pi/2a_{M_1}$, and it follows that

$$\int_{3\pi/2a_{M_1}}^{\pi} G(\theta)\, e^{-in\theta}\, d\theta = O\{\alpha f_A^{-N}(\alpha)\}. \tag{3.4}$$

Equations (3.2), (3.3), and (3.4) give the theorem.

(b) Suppose condition (ii) holds. Let $\mathscr{B}$ be those α for which $\log f_A(\alpha) < N^{-1} \log 1/\alpha$. If $\min \mathscr{B} = 0$, we apply Theorem 3.4, from which it follows that

$$\int_{\alpha f_A{}^{\delta}(\alpha)}^{3\pi/2a_{M_2}} G(\theta)\, e^{-in\theta}\, d\theta = O\{\alpha f_A^{-N}(\alpha)\}. \tag{3.5}$$

as $\alpha \to 0$ through elements of $\mathscr{B}$. If $\alpha \notin \mathscr{B}$, we have $f_A(\alpha) > \alpha^{-1/N}$. With $M_2 = [2N/\delta] + 1$, Eq. (3.5) follows from Theorem 3.3.

Since in case (ii), we assumed that A is a P-sequence and since

$\alpha = O\{f_A^{-1}(\alpha)\}$, it follows from Theorem 3.1 that for $\alpha^{\epsilon/(N+1)} < 3\pi/2a_{M_2}$

$$\int_{3\pi/2a_{M_2}}^{\pi} G(\theta)\, e^{-in\theta}\, d\theta = O\{\alpha f_A^{-(N+)+\epsilon}(\alpha)\}. \tag{3.6}$$

Theorem 3.5 follows when A is a P-sequence from Eqs. (3.2), (3.5) and (3.6) with $\epsilon = \frac{1}{2}$.

We may now complete the proof of Theorem 1.1. If A has property (I), it can easily be shown that $\alpha^2 A_2 = O\{f_A^{1+\epsilon}(\alpha)\}$. Hence Theorem 1.1 follows from Theorems 2.1 and 3.5.

Remarks. The author feels that when $\log a_\nu = O(\log \nu)$, the restriction that A is a P-sequence cannot be relaxed without modifying the theorem. Some preliminary investigations indicate that if the integer p divides all sufficiently large a_ν, then there are significant contributions arising at $\theta_j = j\pi/q$, $j = \pm 1, \pm 2, \ldots, \pm q$. It appears that in some cases, one may only obtain an asymptotic relation for $\log p_A(n)$.

It is easy to show that if A is not a P-sequence, then the assumptions under which Roth and Szekeres obtained asymptotic relations for the number $q_A(n)$ of partitions of n into distinct summands from A are not satisfied. The author has not been able to show this for $p_A(n)$ but feels that it is quite likely so, for the reason in the preceding paragraph.

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