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A Hardy–Ramanujan Formula for Restricted Partitions

GERT ALMKVIST

Mathematics Institute, University of Lund, Box 118, S-221 00 Lund, Sweden

AND

GEORGE E. ANDREWS

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

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In this paper, we extend the Hardy–Ramanujan–Rademacher formula for $p(n)$, the number of partitions of n. In particular we provide such formulas for $p(j, n)$, the number of partitions of j into at most n parts and for $A(j, n, r)$, the number of partitions of j into at most n parts each $\leqslant r$.

1. INTRODUCTION

In this paper we extend the celebrated Hardy–Ramanujan–Rademacher theorem to partitions with restrictions. The new idea is to introduce a differential operator into the formula.

This work was initiated by the first author who wanted to find a practical formula for computing $A(j, n, r)$, the number of partitions of j into at most n parts each $\leqslant r$. $A(j, n, r)$ has a very important application in statistics, the Wilcoxon rank sum test [7]. After months of computations using Fourier series, the saddle point method, and finally Fourier transformations of distributions, he was led to the formula in this paper, but without a correct proof. During this time he benefitted from discussion with Lars Hórmander who also wrote a computer program to check the formula numerically. For this we are most grateful.

During an IMA workshop in Minneapolis in March 1988 the authors met. After a few days the second author proved the formula by induction using Rademacher's exact formula [10, p. 274, Eq. (120.10)] and a result by Hans Peterson [8]. There were, however, some doubts about the

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convergence of a doubly infinite series. These have been avoided in the simplified proof given here.

Finally we want to thank the organizers of the Applied Combinatorics workshop at The Institute for Mathematics and Its Applications, at the University of Minnesota.

2. Background Formulas

To indicate how the formula was found, we first provide a heuristic approach to a result of Glaisher [3] from 1909, wherein the interplay between ξ and $D = d/d\xi$ is crucial.

Theorem 1. *Putting $N = 2j + (n^2 + n)/2$, we have*

$$p(j, n) \approx \frac{1}{(n-1)!} \cdot \left(\prod_{v=1}^{n} \frac{D}{\sinh(vD)} \right) \left(\frac{N}{2} \right)^{n-1}, \tag{2.1}$$

where $D = d/dN$, and $p(j, n)$ is the number of partitions of j into at most n parts.

Remark 1. Expanding the differential operator we obtain Glaisher's formula

$$p(j, n) \approx \frac{1}{n!} \left\{ \frac{1}{(n-1)!} \left(\frac{N}{2} \right)^{n-2} - \frac{s_2}{24(n-3)!} \left(\frac{N}{2} \right)^{n-3} + \frac{5s_2^2 + 2s_4}{5760(n-5)!} \left(\frac{N}{2} \right)^{n-5} - \cdots \right\}, \tag{2.2}$$

where $s_k = \sum_{v=1}^{n} v^k$.

Remark 2. The expression $(d/dN) f(N)$ is shorthand for $((d/dx) f(x))_{x=N}$. This notation was begun by Hardy and Ramanujan [5].

Heuristic Sketch. We have the generating function [2, pp. 3–4]

$$\sum_{j=0}^{\infty} p(j, n)\, t^j = \prod_{v=1}^{n} (1 - t^v)^{-1}. \tag{2.3}$$

Putting $t = e^{i\varphi}$ we have for $\operatorname{Im}(\varphi) > 0$

$$\sum_{j=0}^{\infty} p(j, n)\, e^{ij\varphi} = \left(\frac{i}{\varphi} \right)^n e^{-i(n(n+1)\varphi/4)} \prod_{v=1}^{n} \frac{\varphi/2}{\sin(v\varphi/2)},$$

and we formally compute the Fourier coefficients

$$p(j, n) = \frac{i^n}{2\pi} \int_{-\pi}^{\pi} \left(\prod_{v=1}^{n} \frac{\varphi/2}{\sin(v\varphi/2)} \right) \varphi^{-n}\, e^{-(j + (n^2+n)/4) i\varphi}\, d\varphi.$$

We put $\xi = j + (n^2+n)/4$ and use the notation and results of Hörmander [6, pp. 160, 167]. Thence

$$\int_{-\infty}^{\infty} \varphi^{-n} e^{-i\xi\varphi}\, d\varphi = \frac{\pi}{i^n} \frac{\xi^{n-1}}{(n-1)!}, \qquad \text{since } \xi > 0.$$

Approximating $\int_{-\pi}^{\pi}$ by $\int_{-\infty}^{\infty}$ and using

$$f(\varphi)\, g(\varphi) = f(-\delta)\, \hat{g}(\xi),$$

where

$$\delta = -i \frac{d}{d\xi},$$

we obtain

$$p(j, n) \approx \frac{1}{2(n-1)!} \left(\prod_{\nu=1}^{n} \frac{\delta/2}{\sin(i\nu\delta/2)} \right) \xi^{n-1}$$

which is equivalent to the statement in the theorem. ▮

Remark. We neglected the signularities in $\varphi = \pi/2$, $\varphi = \pi/3$, $2\pi/3$, etc. If these were considered we would obtain Sylvester's 2nd, 3rd, etc. "waves" in the quasi-polynomial $p(j, n)$.

Next we state a result from [1] (see also [12]) for $A(j, n, r)$ when j is close to $nr/2$. These values of j are the middle values since $A(j, n, r) = A(nr - j, n, r)$ for $j = 0, 1, \dots, nr$. Also since $A(j, n, r) = A(n, j, r)$ we restrict our attention to j.

THEOREM 2. *Put $p = n + r + 1$ and $v = \sqrt{(12/npr)}(j - nr/2)$. Then*

$$A(j, n, r) \approx \binom{n+r}{r} \sqrt{\frac{6}{\pi npr}}\, e^{-v^2/2} \left\{ 1 - \frac{1}{20} \left(\frac{1}{n} + \frac{1}{r} - \frac{1}{p} \right) (3 - 6v^2 + v^4) \right\}. \quad (2.4)$$

COROLLARY. *Let $c(n, r)$ be the number of $SL(2, \mathbb{C})$-invariants of degree r of a binary form of degree n. Then by Cayley–Sylvester* [11, p. 65]

$$c(n, r) = A\left(\frac{nr}{2}, n, r\right) - A\left(\frac{nr}{2} - 1, n, r\right) \approx \frac{1}{\sqrt{\pi}} \binom{n+r}{r} \left(\frac{6}{npr}\right)^{3/2}. \quad (2.5)$$

We note that in [1, 12] better approximations are given.

3. The Main Results

THEOREM 3. *Let $f(x)=\sum_{j=0}^{\infty} a_j x^j$ be a polynomial. Define*

$$S_N(x)=\prod_{\nu=1}^{N-1}(1-x^\nu) \tag{3.1}$$

and

$$z=z_{p,q}=e^{-2(D-ip\pi/q)}, \tag{3.2}$$

where $D=d/d\xi$ and $\xi=2j-1/12$. Then

$$a_j=\frac{4\sqrt{3}}{\pi^2}\sum_{q=1}^{\infty}\sum_{(p,q)=1} q^{3/2}\omega_{p,q}\, e^{-2j(p\pi i/q)} f(z)\, S_j(z)\, D^2 \cosh\frac{\pi}{q}\sqrt{\xi/3}, \tag{3.3}$$

where ω_{pq} is a certain 24qth *root of unity* [2, p. 71, Eq. (5.2.4)].

Proof. By additivity it is enough to prove the theorem for the function

$$f(z)=z^\lambda,$$

where $\lambda\in\mathbb{N}$. From Euler's pentagonal number theorem [2, p. 11] we have

$$S_j(z)\equiv 1+\sum_{k(3k-1)/2<j}(-1)^k(z^{k(3k-1)/2}+z^{k(3k+1)/2}) \qquad (\mathrm{mod}\ z^j) \tag{3.4}$$

(considered in the formal power series ring $\mathbb{C}[\![z]\!]$).

Now let $p^*(j)$ denote the right hand side in the Hardy–Ramanujan–Rademacher formula [10, p. 274, Eq. (120.10)], i.e.,

$$p^*(j)\equiv\frac{4\sqrt{3}}{\pi^2}\sum_{q=1}^{\infty}\sum_{(p,q)=1} q^{3/2}\omega_{p,q}\, e^{-2jp\pi i/q}\, D^2\cosh\frac{\pi}{q}\sqrt{\xi/3}. \tag{3.5}$$

Then for all $j\in Z$

$$p^*(j)=\begin{cases} p(j) & \text{if } j>0\\ 1 & \text{if } j=0\\ 0 & \text{if } j<0,\end{cases} \tag{3.6}$$

where the last line is due to Hans Peterson [8] (see also Rademacher [9, p. 71]). Let

$$g_N(z)=z^\lambda+\sum_{k=1}^{N}(-1)^k\{z^{\lambda+(k(3k-1)/2)}+z^{\lambda+(k(3k+1)/2)}\},$$

where N is to be determined later. Then

$$\begin{aligned}
&\frac{4\sqrt{3}}{\pi^2}\sum_{q=1}^{\infty}\sum_{(p,q)=1} q^{3/2}\omega_{p,q}\, e^{-2jp\pi i/q} g_N(z)\, D^2 \cosh\frac{\pi}{q}\sqrt{\xi/3} \\
&\quad = p^*(j-\lambda)+\sum_{k=1}^{N}(-1)^k\left\{p^*\left(j-\lambda-\frac{k(3k-1)}{2}\right)\right. \\
&\qquad \left. + p^*\left(j-\lambda-\frac{k(3k+1)}{2}\right)\right\} \\
&\quad = \begin{cases} 1 & \text{if } j=\lambda \\ 0 & \text{otherwise,}\end{cases}
\end{aligned} \tag{3.7}$$

provided we choose N such that

$$j-\lambda-\frac{N(3N-1)}{2}<0,$$

e.g., $N\geqslant 1+\sqrt{2(j-\lambda)/3}$.

The second equality follows from (3.6). Note that it is legitimate to sum over q first since we have only a finite sum over k and the sum on q is absolutely convergent. To show the first equality it is enough to show

$$\frac{4\sqrt{3}}{\pi^2}\sum_{q=1}^{\infty}\sum_{(p,q)=1} q^{3/2}\omega_{p,q}\, e^{-2jp\pi i/q} z^s D^2 \cosh\frac{\pi}{q}\sqrt{\xi/3}=p^*(j-s). \tag{3.8}$$

But this follows immediately from Taylor's formula

$$e^{-2sS}h(\xi)=h(\xi-2s)$$

and the definition of p^*.

To complete the proof we only have to note that

$$g_N(z)\equiv z^\lambda S_j(z) \qquad (\mathrm{mod}\ z^{j+\lambda})$$

for all $\lambda\geqslant 0$ if N is chosen as above. ▌

As our first application of Theorem 3, we take

$$\begin{aligned}
f(x) &= \begin{bmatrix} n+r \\ r \end{bmatrix}_x \\
&\equiv \prod_{\nu=1}^{r}\frac{1-x^{n+\nu}}{1-x^{\nu}} \\
&= \sum_{j=0}^{nr} A(j,n,r)x^j,
\end{aligned} \tag{3.9}$$

where $A(j, n, r)$ was defined in Section 1.

THEOREM 4.

$$A(j,n,r)=\frac{4\sqrt{3}}{\pi^2}\sum_{q=1}^{\infty}\sum_{(p,q)=1} q^{3/2}\omega_{p,q}\,e^{-2jp\pi i/q}\begin{bmatrix} n+r \\ r\end{bmatrix}_z S_j(z)$$
$$\times D^2\cosh\frac{\pi}{q}\sqrt{\xi/3}. \tag{3.10}$$

If we choose $r\geqslant j$, we obtain an asymptotic formula for $p(n,j)$.

THEOREM 5.

$$p(j,n)=\frac{4\sqrt{3}}{\pi}\sum_{q=1}^{\infty}\sum_{(p,q)=1} q^{3/2}\omega_{p,q}\,e^{-2jp\pi i/q}$$
$$\times\prod_{v=n+1}^{j-1}(1-z^v)\times D^2\cosh\frac{\pi}{q}\sqrt{\xi/3}. \tag{3.11}$$

4. COMPUTATION OF $p(j,n)$

In this section we shall examine in detail how (3.11) can actually be used to compute effectively $p(j,n)$.

We note by [2, p. 19]

$$\begin{aligned} R_{j,n}(x) &= \prod_{v=n+1}^{j-1}(1-x^v) \\ &\equiv \prod_{\mu=0}^{\infty}(1-x^{n+1+\mu}) \qquad (\mathrm{mod}\ x^j) \\ &\equiv 1+\sum_{k=1}^{N}\frac{(-1)^k x^{k(k+1)/2+kn}}{(1-x)(1-x^2)\cdots(1-x^k)} \qquad (\mathrm{mod}\ x^j) \end{aligned} \tag{4.1}$$

provided we choose N so that $N(N+1)/2+Nn\geqslant j-1$.

Consequently for our differential operators (mod z^j)

$$R_{j,n}(z)=1+\sum_{k=1}^{N}(-1)^k z^{k(k+1)/2+kn}\prod_{\mu=1}^{k}\left(1+\coth\mu\left(D-\frac{ip\pi}{q}\right)\right). \tag{4.2}$$

We now rewrite Theorem 5 in the form

$$p(j,n)=\sum_{q=1}^{\infty}\Phi_q(j,n), \tag{4.3}$$

where

$$\Phi_q(j,n)=\frac{4\sqrt{3}}{\pi^2}\,q^{3/2}\sum_{(p,q)=1}\omega_{p,q}\,e^{-2jp\pi i/q}R_{j,n}(z)\,D^2\cosh\frac{\pi}{q}\sqrt{\xi/3}. \tag{4.4}$$

Then we have

$$\begin{aligned}\Phi_1(j,n)&=\frac{4\sqrt{3}}{\pi^2}\left\{1+\sum_{k=1}^{N}\frac{(-1)^k}{2^k}e^{-k(2n+k+1)D}\right.\\&\qquad\left.\times(1+\coth D)\cdots(1+\coth kD)\right\}D^2\cosh\pi\sqrt{\xi/3}\\&=\frac{4\sqrt{3}}{\pi^2}\left\{D^2-e^{-(2n+2)D}\left(\frac{D}{2}+\frac{D^2}{2}+\frac{D^3}{6}-\frac{D^5}{90}+\frac{D^7}{945}-\cdots\right)\right.\\&\qquad+e^{-(4n+6)D}\left(\frac{1}{8}+\frac{3D}{8}+\frac{11}{24}D^2+\frac{D^3}{4}+\frac{D^4}{120}+\cdots\right)\\&\qquad-e^{-(6n+12)D}\left(\frac{1}{48D}+\frac{1}{8}+\frac{47}{144}D+\frac{11}{24}D^2+\cdots\right)\\&\qquad\left.+e^{-(8n+20)D}\left(\frac{1}{384D^2}+\frac{5}{192D}+\frac{133}{1152}+\cdots\right)+\cdots\right\}\cosh\pi\sqrt{\xi/3}.\end{aligned}$$

Here D^2 gives the first term in the Hardy–Ramanujan formula. The advantage with the formula for Φ_1 is that

$$e^{-\nu D}h(\xi)=h(\xi-\nu)$$

by Taylor's fromula. If $k(2n+k+1)>2j-1/12$ then sinh (resp. cosh) is changed into sin (resp. cos) and we neglect these terms.

L. Hörmander has written a program in APL that computes Φ_1. In practice we replace $\cosh\pi\sqrt{\xi/3}$ by $\frac{1}{2}\exp(\pi\sqrt{\xi/3})$ and use the formula

$$D^k e^{a\sqrt{\xi}}=b^kQ_k(z)\,e^{a\sqrt{\xi}}\qquad(k\in\mathbb{Z})$$

with $z=1/(a\sqrt{\xi})$ and $b=a/(2\sqrt{\xi})$. The Q_k are polynomials satisfying $Q_0=Q_1=1$ and

$$Q_{k+1}=Q_{k-1}-(2k-1)\,zQ_k$$

(also valid for negative k). The Q_k are related to the modified Bessel functions $I_{-k-1/2}$.

We now turn to the higher order terms Φ_q for $q\geqslant 2$. It is easily shown that they are real (it follows from $\omega_{q-p,p}=\bar{\omega}_{p,q}$).

$q=2$.

We have

$$1+\coth k\left(D-\frac{i\pi}{2}\right)=\begin{cases}1+\coth kD, & k \text{ even}\\ 1+\tanh kD, & k \text{ odd.}\end{cases}$$

It follows

$$\begin{aligned}\Phi_2(j,n)&\\ =(-1)^j\frac{8\sqrt{6}}{\pi^2}&\left\{D^2+(-1)^n\left[e^{-(2n+2)D}\left(\frac{D^2}{2}+\frac{D^3}{2}-\frac{D^5}{6}+\frac{D^7}{15}+\cdots\right)\right.\right.\\ &-e^{-(4n+6)D}\left(\frac{D}{8}+\frac{3D^2}{8}+\frac{5}{12}D^3+\frac{D^4}{8}-\frac{23}{180}D^5-\frac{D^6}{12}+\frac{19}{378}D^7+\cdots\right)\\ &\left.\left.+e^{-(6n+12)D}\left(\frac{D}{16}+\frac{3}{8}D^2+\frac{37}{48}D^3+\frac{D^4}{8}+\cdots\right)\right]+\cdots\right\}\cosh\frac{\pi}{2}\sqrt{\xi/3}.\end{aligned}$$

Here it can occur that the third term is larger than the second.

$q=3$.

We obtain

$$\begin{aligned}\Phi_3(j,n)=\frac{36}{\pi^2}&\left\{D^2-(A_3(j-n-1)\,e^{-2nD}-A_3(j-n)\,e^{-(2n+2)D})\right.\\ &\times\left(\frac{D^2}{3}-\frac{4D^4}{9}+\frac{4D^6}{9}+\cdots\right)+[A_3(j-2n)(e^{-4nD}+e^{-(4n+6)D})\\ &-A_3(j-2n-2)\,e^{-(4n+2)D}-A_3(j-2n-1)\,e^{-(4n+4)D}]\\ &\left.\times\left(\frac{D^2}{9}-\frac{20}{27}D^4+\frac{268}{81}D^6+\cdots\right)-\cdots\right\}\cosh\frac{\pi}{3}\sqrt{\xi//3}\end{aligned}$$

where

$$A_3(k)=2\cos\left(\frac{2k\pi}{3}-\frac{\pi}{18}\right).$$

$q=4$.

$$\begin{aligned}\Phi_4(j,n)=\frac{32\sqrt{3}}{\pi^2}&\left\{D^2-[A_4(j-n-1)\,e^{-2nD}-A_4(j-n)\,e^{-(2n+2)D}]\right.\\ &\left.\times\left(\frac{D^2}{2}-D^4+\frac{5}{48}D^6+\cdots\right)+\cdots\right\}\cosh\frac{\pi}{4}\sqrt{\xi/3}\end{aligned}$$

where

$$A_4(k) = 2\cos\left(\frac{k\pi}{2} - \frac{\pi}{8}\right).$$

Finally we take a numerical example to show the size of the different terms.

EXAMPLE 9. Let $j = 200$ and $n = 100$. We compute

$$p(j) - p(j, n) = \sum_q \tilde{\Phi}_q(j, n).$$

Here $\tilde{\Phi}_q$ is Φ_q with the first term deleted and the signs changed for the other terms.

Then we have

$\tilde{\Phi}_1 =$	14524	23443.0133
$\tilde{\Phi}_2 =$		-169.6790
$\tilde{\Phi}_3 =$		3.3310
$\tilde{\Phi}_4 =$		-0.4773
$\tilde{\Phi}_5 =$		-0.1900
	14524	23275.9980

Now

$$p(200) - p(200, 100) = 3972999029388 - 3971546606112 = 1452423276$$

so the error is 0.0020.

REFERENCES

1. G. ALMKVIST, How to compute the number of restricted partitions, preprint, Lund, 1987.
2. G. E. ANDREWS, "The Theory of Partitions," Addision–Wesley, Reading, 1976. (Reissued by Cambridge University Press, Cambridge, 1985).
3. J. W. L. GLAISHER, Formulae for partitions into given elements derived from Sylvester's theorem, *Quart. J. Math.* **40** (1909), 275–348.
4. G. H. HARDY AND S. RAMANUJAN, Une formule asymptotique pour le nombre des partitions de *n*, *Comptes rendus* **164** (1917), 35–38.
5. G. H. HARDY AND S. RAMANUJAN, Asymptotic formulae in combinatory analysis, *Proc. London Math. Soc.* **17** (1918), 75–115.
6. L. HÖRMANDER, "The Analysis of Linear Partial Differential Operators," Vol. 1, Springer-Verlag, New York, 1983.

7. H. B. MANN AND D. R. WHITNEY, On a test of whether one or two random variables is stochastically larger than the other, *Ann. Math. Stat.* **18** (1947), 50–60.
8. H. PETERSON, Die linearen Relationen zwischen den ganzen Poincaréschen Reihe von reller Dimension zur Modulgruppe, *Abh. Math. Sem. Univ. Hamburg.* **12** (1938), 415–472.
9. H. RADEMACHER, Fourier expansions of modular forms and problems of partition, *Bull. Amer. Math. Soc.* **46** (1940), 59–73. (Reprinted *in* "Collected Papers of H. Rademacher," Vol. II, pp. 204–219, MIT Press, Cambridge, 1974.)
10. H. RADEMACHER, "Topics in Analytic Number Theory," Springer-Verlag, New York, 1973.
11. I. SCHUR, "Vorlesungen über Invariantentheorie," Springer-Verlag, Berlin, 1968.
12. L. TAKÁCS, Some asymptotic formulas for lattice paths, *J. Statist. Plann. Inference* **14** (1986), 123–142.