# Avalanche Collapse of Interdependent Networks

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We reveal the nature of the avalanche collapse of the giant viable component in multiplex networks under perturbations such as random damage. Specifically, we identify latent critical clusters associated with the avalanches of damage. Divergence of their mean size signals the approach to the hybrid phase transition from one side, while there are no critical precursors on the other side. We find that this discontinuous transition occurs in scale-free multiplex networks whenever the mean degree of at least one of the interdependent networks does not diverge.

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Many complex systems, both natural [1], and man-made [2, 3], can be represented as multiplex or interdependent networks. Multiple dependencies make a system more fragile: damage to one element can lead to avalanches of failures throughout the system [4, 5]. Recent theoretical investigation of two [6] or more [7] networks in which vertices in each network mutually depend on vertices in other networks has shown that indeed small initial failures can cascade back and forth through the networks, leading to a discontinuous collapse of the whole system. It was shown in [8] that there is a simple mapping between the model used in [6] in which a vertex in one network has a mutual dependence on exactly one vertex in the other network, and a multiplex network with one kind of vertex but two kinds of edges. The mapping is achieved by simply merging the mutually dependent vertices from the two networks.

In this Letter we describe the nature of such transitions. We consider a set of vertices connected by $m$ different types of edges (dependencies). The connections are essential to the function of each site, so that a vertex is only viable if it maintains connections of every type to other viable vertices. A *viable cluster* is defined as follows: For every kind of edge, and for any two vertices $i$ and $j$ within a viable cluster, there must be a path from $i$ to $j$ following only edges of that kind. A graph containing two finite viable clusters is illustrated in Fig. 1. We wish to find when there is a giant cluster of viable vertices. Note that any giant viable cluster is a subgraph of the giant connected component of each of the $m$ networks.

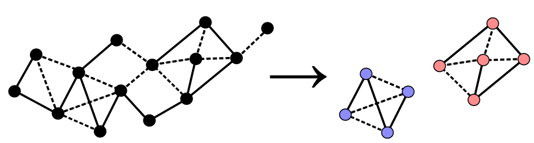

FIG. 1: A small network with two kinds of edges (left). Applying the algorithm described in the text non-viable vertices are removed leaving two viable clusters (right).

Various parameters can be used to control the critical behavior of this system: the mean degrees of the networks, amount of random damage and so on. Small perturbations to the system can propagate, leading to avalanches of further damage. In uncorrelated, random networks we find a discontinuous hybrid transition in the collapse of the giant viable cluster, similar to that seen in the $k$-core or bootstrap percolation [9, 10]. Avalanches of removals of vertices from the giant viable cluster increase in size approaching the critical point, signaling the impending collapse of the giant viable cluster. At the critical point the mean avalanche size diverges. Below the transition, on the other hand, there is no precursor for the appearance of the giant viable cluster. The transition is thus asymmetric. It is hybrid in nature, having a discontinuity like a first-order transition, but exhibiting critical behavior, only above the transition, like a second-order transition. A complete understanding of the transition cannot therefore be had without first understanding this critical behavior. We have discovered critical clusters which collapse in avalanches of diverging size as the transition is approached. These critical clusters are thus responsible for both the critical scaling and the discontinuity observed in the size of the giant viable cluster. The critical clusters have a novel character as, unlike the corona clusters of the $k$-core for example [9], Avalanches propagate in a directed way through critical clusters. The critical clusters may have important practical applications, helping to identify vulnerabilities to targeted attack, as well as informing efforts to guard against such attack. Surprisingly, when the degree distributions are asymptotically power-law $P(q) \propto q^{-\gamma}$ the critical point $p_c$ (taking undamaged fraction of vertices $p$ as the control parameter) remains at a finite value even when the exponents $\gamma$ of the degree distributions are below 3, remaining finite until both exponents reach 2. This is in stark contrast to ordinary percolation, in which the threshold falls to zero as soon as $\gamma$ reaches 3. Furthermore, the nature of the transition doesn't change. Although the height of the discontinuity becomes extremely small near $\gamma = 2$, it remains finite near this limit (see Fig. 2).

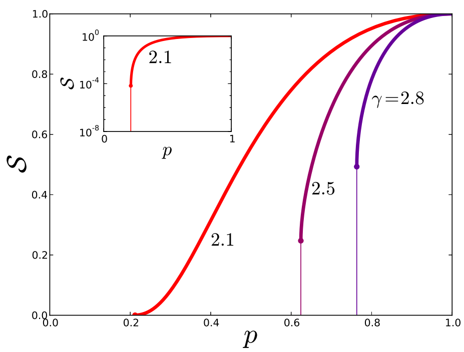


FIG. 2: Size of the giant viable cluster $\mathcal{S}$ as a function of the fraction $p$ of vertices remaining undamaged for two symmetric powerlaw distributed networks with, from right to left, $\gamma = 2.8$ , 2.5, and 2.1. The height of the jump becomes very small as $\gamma$ approaches 2, but is not zero, as seen in the inset, which is $\mathcal{S}$ vs $p$ on a logarithmic vertical scale for $\gamma = 2.1$.

*Algorithm.*—We consider a multiplex network, with vertices $i = 1, 2, ..., N$ connected by $m$ kinds of vertices labeled $s = a, b, ....$ The joint degree distribution is $P(q_a, q_b, ...)$. Viable clusters in any multiplex network may be identified by the following algorithm.
(i) Choose a test vertex $i$ at random from the network.
(ii) For each kind of edge $s$, compile a list of vertices that can be reached from $i$ by following only edges of type $s$.
(iii) The intersection of these $m$ lists forms a new candidate set for the viable cluster containing $i$.
(iv) Repeat steps (ii) and (iii) but traversing only the current candidate set. When the candidate set reaches an equilibrium, it is either a viable cluster, or contains only vertex $i$.
(v) To find further viable clusters, remove the viable cluster of $i$ from the network (cutting any edges) and repeat steps (i)-(iv) on the remaining network beginning from a new test vertex.

Repeated application of this procedure will identify every viable cluster in the network. The application of this procedure to a finite graph is illustrated in Fig. 1.

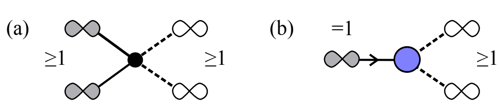


FIG. 3: Viable and critical viable vertices for two interdependent networks. (a) A vertex is in the giant viable cluster if it has connections to giant viable subtrees (represented by infinity symbols) of both kinds. (b) A critical viable vertex of type $a$ has exactly 1 connection to a giant sub-tree of type $a$.

*Basic Equations.*—Let us consider the case of sparse uncorrelated networks, which are locally tree-like in the infinite size limit $N \to \infty$. In such a network there are no finite clusters. In order to find the giant viable cluster, let us define $X_s$, with $s \in \{a, b, ...\}$, to be the probability

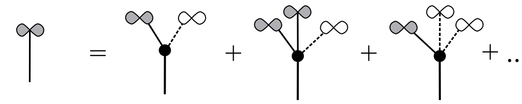

FIG. 4: Diagrammatic representation of Eq. (1) in a system of two interdependent networks $a$ and $b$. The probability $X_a$, represented by a shaded infinity symbol can be written recursively as a sum of second-neighbor probabilities. Open infinity symbols represent the equivalent probability $X_b$ for network $b$, which obeys a similar recursive equation.

that, on following an arbitrarily chosen edge of type $s$, we encounter the root of an infinite sub-tree formed solely from type $s$ edges, whose vertices are also each connected to at least on infinite subtree of every other type. We call this a type $s$ infinite subtree. The vector $\{X_a, X_b, ...\}$ plays the role of the order parameter. A vertex is then in the giant viable cluster if it has at least one edge of every type $s$ leading to an infinite type $s$ sub-tree (probability $X_s$), as shown in Fig. 3(a). Using the locally tree-like property of the networks, we can write self consistency equations for the probabilities $X_s$:

$$X_s = \Psi_s(X_a, X_b, ...) \equiv \sum_{q_a,q_b,...} \frac{q_s}{\langle q_s \rangle} P(q_a, q_b, ...) \big[1-(1-X_s)^{q_s-1}\big] \prod_{l \neq s} \big[1-(1-X_l)^{q_l}\big] \tag{1}$$

for each $s \in \{a, b, ...\}$. This is illustrated in Fig. 4. The term $(q_s/\langle q_s \rangle)P(q_a, q_b, ...)$ gives the probability that on following an arbitrary edge of type $s$, we find a vertex with degrees $q_a, q_b, ...$, while $[1 - (1-X_a)^{q_a}]$ is the probability that this vertex has at least one edge of type $a \neq s$ leading to the root of an infinite sub-tree of type $a$ edges (i.e. probability $X_a$). This becomes $[1 - (1-X_s)^{q_s-1}]$ when $a = s$. Solving these equations enables us to calculate all the quantities of interest. In particular, the relative size of the giant viable cluster is given by

$$\mathcal{S} = \sum_{q_a,q_b,...} P(q_a, q_b, ...) \prod_{s=a,b,...} \big[1 - (1-X_s)^{q_s}\big], \tag{2}$$

which is illustrated in Fig. 3(a).

A hybrid transition appears at the point where $\Psi_s(X_a, X_b, ...)$ first meets $X_s$ at a non-zero value, for all $s$. This occurs when

$$\det[\mathbf{J} - \mathbf{I}] = 0 \tag{3}$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{J}$ is the Jacobian matrix $J_{ab} = \partial\Psi_b/\partial X_a$. Expanding $\Psi_s$ about the critical point, at which Eqs. (1) and (3) are both satisfied, we find the scaling of $X_s$ and hence $\mathcal{S}$, the size of the giant viable cluster. For example, to consider random damage, we introduce a factor of $p$ to Eqs. (1) and (2). The control variable $p$ is the fraction of vertices remaining undamaged. Then

$$\mathcal{S} - \mathcal{S}_c \propto X_s - X_s^{(c)} \propto (p - p_c)^{1/2}. \tag{4}$$

A similar result is found for other control parameters.

*Avalanches.*—To examine the hybrid transition we focus on the case of two types of edges. Consider a viable vertex that has exactly one edge of type $a$ leading to a type $a$ infinite subtree, and at least one edge of type $b$ leading type $b$ infinite subtrees. We call this a critical vertex of type $a$. It is illustrated in Fig. 3(b). Each critical vertex has one special edge on which it depends to remain viable; critical vertices of type $a$ will drop out of the viable cluster if they lose their single link to a type $a$ infinite subtree. We mark these special edges with an arrow leading to the critical vertex. An avalanche can only transmit in the direction of the arrows. A vertex may have outgoing edges of this kind, so that removal of this vertex from the giant viable cluster also requires the removal of the critical vertices which depend on it. For example, in Fig. 5, removal of the vertex labeled 1 removes the essential edge of the critical vertex 2 which thus becomes non-viable. Removed critical viable vertices may in turn have outgoing critical edges, so that the removal of a single vertex can result in an avalanche of removals of critical vertices from the giant viable cluster. In Fig. 5, removal of 2 causes the removal of further critical vertices 3 and 4, and the removal of 4 then requires the removal of 5. Thus critical vertices form critical clusters. At the head of each critical cluster is a 'keystone vertex' (e.g. vertex 1 in the figure) whose removal would result in the removal of the entire cluster. Graphically, upon removal of a vertex, we remove all vertices found by following the arrowed edges. As we approach the critical point (from above), the mean size of the critical clusters diverges. The avalanches cause a discontinuity in the size of the giant viable cluster, which collapses to zero.

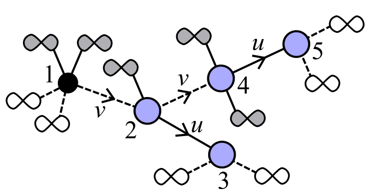


FIG. 5: A critical cluster. Removal of any of the shown viable vertices will result in the removal of all downstream critical viable vertices. Removal of the vertex labeled 1 will result in all of the shown vertices being removed (becoming non-viable), while removal of vertex 4 results only in vertex 5 also being removed.

There are three possibilities when following an arbitrarily chosen edge of a given type: i) with probability $X_s$ we encounter a type $s$ infinite subtree ii) with probability $R_s$ we encounter a vertex which has a connection to an infinite subtree of the opposite type, but none of the same type. Such a vertex is part of the giant viable cluster if the parent vertex was; or iii) with probability $1-X_s-R_s$, we encounter a vertex which has no connections to infinite subtrees of either kind. The probability $R_a$ obeys

$$R_a = \sum_{q_a}\sum_{q_b}\frac{q_a}{\langle q_a\rangle}P(q_a,q_b)(1-X_a)^{q_a-1}\left[1-(1-X_b)^{q_b}\right] \tag{5}$$

and similarly for $R_b$. We use generating functions to examine the sizes of critical clusters. We first define the function $F_a(x,y)$ as

$$F_a(x,y) = \sum_{q_a}\sum_{q_b}\frac{q_a}{\langle q_a\rangle}P(q_a,q_b)x^{q_a-1}\sum_{r=1}^{q_b}\binom{q_b}{r}X_b^r y^{q_b-r} \tag{6}$$

and similarly for $F_b(x,y)$, by exchanging all subscripts $a$ and $b$. The generating function for the size of a critical cluster reached upon following an arbitrary type $a$ edge which does not lead to an infinite type $a$ subtree can be defined in terms of these functions by

$$H_a(u,v) = 1 - X_a - R_a + uF_a[H_a(u,v),H_b(u,v)] \tag{7}$$

and similarly for $H_b(u,v)$. This recursive equation can be understood by noting that $H_a(0,v) = 1 - X_a - R_a$ is the probability that an arbitrarily chosen edge leads to a 'dead-end', i.e. a vertex outside the viable cluster. Here $u$ and $v$ are auxiliary variables. Following through a critical cluster, a factor $u$ appears for each arrowed edge of type $a$, and $v$ for each arrowed edge of type $b$. For example, the critical cluster illustrated in Fig. 5 contributes a factor $u^2v^2$. The mean number of critical vertices reached upon following an edge of type $a$, i.e. the mean size of the resulting avalanche if this edge is removed, is given by $\partial_u H_a(1,1) + \partial_v H_a(1,1)$. Unbounded avalanches emerge at the point where $\partial_u H_a(1,1)$ [or $\partial_v H_b(1,1)$] diverges. Taking derivatives of Eq. (7), and using that $H_a(1,1) = 1-X_a$ and $F_a(1-X_a,1-X_b) = R_a$, and that, from Eqs. (1) and (6), $\partial_x F_a(1-X_a,1-X_b) = \partial_a\Psi_a(X_a,X_b)$ and $\partial_y F_a(1-X_a,1-X_b) = (\langle q_a\rangle/\langle q_b\rangle)\partial_a\Psi_b(X_a,X_b)$, gives

$$\partial_u H_a(1,1) = \frac{R_a[1-\partial_b\Psi_b(X_a,X_b)]}{\det[\mathbf{J}-\mathbf{I}]}. \tag{8}$$

From Eq. (3) we see immediately that this diverges at the critical point, meaning that the mean avalanche size diverges exactly at the point of the hybrid transition (i.e. there is only one critical point).

*Scale-free Networks.*—In ordinary percolation, and even the $k$-core and heterogeneous $k$-core, networks with degree distributions that are asymptotically powerlaws $P(q) \sim q^{-\gamma}$ may exhibit qualitatively different transitions, especially when $\gamma < 3$. To investigate such effects in the giant viable cluster, we consider powerlaw degree distributions with fixed minimum degree $q_0 = 1$ (then $\langle q\rangle \approx (\gamma-1)q_0/(\gamma-2)$), so that $P(q_s) = \zeta(\gamma_s)q^{-\gamma_s}$ where $s$ takes the values $a$ or $b$. As a control parameter we apply random damage to the system as a whole so that vertices survive with probability $p$. First consider the case where either $\gamma_a$ or $\gamma_b$, or both, is greater than 3. The giant viable cluster is necessarily a subgraph of the overlap between the giant-components of each graph. We know from ordinary percolation that for $\gamma > 3$, the giant component appears at a finite value of $p$ [11]. It follows that the giant viable cluster, also, cannot appear from $p = 0$; there must be a finite threshold $p_c$, (with a hybrid transition) as for other networks such as Erdős-Rényi. This is true even if one of the networks has $\gamma_s < 3$.

The more interesting case is when both degree exponents $\gamma_s$ are less than three, when the percolation threshold is zero for each network when considered separately. Let us write $\gamma_a = 2+\delta_a$ and $\gamma_b = 2+\delta_b$, and examine the behavior for small $\delta_a$ and $\delta_b$. We proceed by assuming that in this situation, for $p$ near $p_c$, Eqs. (1) have a solution with small $X_a$ and $X_b \ll 1$. Writing only leading orders of $X_a$ and $X_b$, and $\delta_a$ and $\delta_b$, we find that

$$\Psi_a(X_a, X_b) = p\frac{\pi^2}{6\,\delta_b} X_a^{\delta_a} \left( X_b - X_b^{1+\delta_b} \right) \tag{9}$$

and similarly for $\Psi_b(X_a, X_b)$. The location of the critical point is found from Eq. (3) which becomes

$$\delta_a + \delta_b = p\frac{\pi^2}{6} X_a^{\delta_a} X_b^{\delta_b} \left( \frac{X_a}{X_b} + \frac{X_b}{X_a} \right) . \tag{10}$$

Solving Eqs. (9) and (10) and using Eq. (2) allows us to calculate $X_s$ and $\mathcal{S}$ at $p_c$. We find in general that the hybrid transition persists as long as $\delta_a$ and $\delta_b$ are not zero, though the height of the discontinuity at the hybrid transition becomes extremely small for $\delta$ small. In experiments or simulations, this could be misinterpreted as evidence of a continuous phase transition. We here describe two representative cases. First, where $\delta_a \ll \delta_b$, that is, $\gamma_a$ tends to 2 while $\gamma_b$ stays a little larger than 2. We find that the location of the discontinuous transition is $p_c \approx 1.19\delta_b$, and the size of the giant viable cluster at the critical point is $S_c = Ae^{-B/\delta_b}$ with $A \approx 3.36$ and $B \approx 2.89$. We see that a hybrid transition occurs, albeit with an extremely small discontinuity, at a nonzero threshold $p_c$ as long as at least one of $\delta_a$ and $\delta_b$ is not equal to zero. To examine the case that both tend to zero, we consider the symmetric case $\delta_a = \delta_b \equiv \delta$. Then $X_a = X_b \equiv X$, and the discontinuity is found by requiring $\Psi'(X) = 1$ [from Eq. (3)]. We find that $X_c = (1/2)^{1/\delta}$, $p_c = 24\delta/\pi^2$, and, $\mathcal{S}_c = 4^{1-1/\delta}$. The location of the hybrid transition tends to $p = 0$ as $\delta \to 0$, and the size of the 'jump' becomes very small even for nonzero $\delta$, but vanishes completely as $\delta \to 0$. In Fig. 2 we plot the size of the giant viable cluster in this symmetric case for three values of $\gamma$. For values not close to 2, the transition looks similar to that observed in, say, Erdős-Rényi graphs. As $\gamma$ approaches 2, however, we see that the height of the discontinuity becomes extremely small. Nevertheless, the square root scaling and non-zero critical point are retained. Expanding $\Psi(X)$ about $X_c$ we find that

$$\frac{X - X_c}{X_c} = \frac{12}{\pi^2\,\delta\,p_c} \left( \frac{p - p_c}{p_c} \right)^{1/2} \tag{11}$$

which holds so long as $p - p_c \ll \delta^3$. That is, the scaling is square-root in a narrow region of width $\mathcal{O}(\delta^3)$ above the hybrid transition. This region disappears as $\delta \to 0$.

*Summary.*—We have given an algorithm for identifying the viable clusters in any multiplex network. Under increasing damage, the giant viable cluster collapses in a discontinuous hybrid transition, in contrast to the smooth continuous transition found in simplex networks. We have shown that this transition is signaled by avalanches whose mean size diverges as the collapse approaches. To understand this critical behavior, which occurs only above the transition, we successfully identified clusters of critical vertices. These clusters determine the structure and statistics of avalanches of damage. Avalanches sweep through the critical clusters in a directed fashion, and it is the diverging size of these clusters which accounts for the criticality. This directed nature stands in contrast to, for example, the corona clusters found in the $k$-core problem [12]. Each critical cluster depends upon a keystone vertex whose removal completely destroys the critical cluster. These keystone vertices are good candidates for targeted attack or immunization against such attacks.

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