From number theory to statistical mechanics: Bose–Einstein condensation in isolated traps

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Abstract

We question the validity of the grand canonical ensemble for the description of Bose–Einstein condensation of small ideal Bose gas samples in isolated harmonic traps. While the ground state fraction and the specific heat capacity can be well approximated with the help of the conventional grand canonical arguments, the calculation of the fluctuation of the number of particles contained in the condensate requires a microcanonical approach. Resorting to the theory of restricted partitions of integer numbers, we present analytical and numerical results for such fluctuations in one– and three-dimensional traps, and show that their magnitude is essentially independent of the total particle number.

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I. BOSE–EINSTEIN CONDENSATION IN ISOLATED TRAPS: CAN WE RELY ON THE GRAND CANONICAL ENSEMBLE?

After the breakthrough in the preparation and detection of Bose–Einstein condensates of rubidium [1], sodium [2], and lithium [3], the second generation of experiments on ultracold, magnetically trapped Bose gases now starts to probe condensate properties such as the temperature dependence of the condensate fraction [4,5] or collective excitations of the condensate [6–8]. A matter-of-principle experiment has shown the feasibility of an output coupler for condensed trapped atoms [9], two different, overlapping condensates have been produced in a single trap [10], and interference fringes between two freely expanding condensates have been demonstrated [11], thus proving first-order coherence of "mesoscopic" matter waves. Higher-order coherence has been inferred from three-body recombination rates of condensed atoms [12], and even the propagation of sound in a Bose condensate could be directly observed [13].

These developments naturally force the theorist to investigate a conceptually important question: When dealing with Bose–Einstein condensates consisting of about 10^3 to 10^6 particles in a magnetic trap, to which extent can the different thermodynamical ensembles be considered as equivalent? A Bose gas cooled below the condensation point and kept in a trap is neither in thermal contact with a heat bath, nor exchanging particles with some particle reservoir. Hence, the only ensemble that is appropriate for describing this situation is the microcanonical one. And yet, the theoretical discussion of trapped condensates so far mainly relies on the grand canonical ensemble.

Within the grand canonical ensemble, the description of a trapped, ideal Bose gas consisting of N particles is comparatively simple. We consider an isotropic harmonic trap with oscillator frequency ω, so that the degree of degeneracy of the j-th single-particle state is

$$g_j = \frac{1}{2}(j+1)(j+2) \; . \tag{1}$$

Hence, the difference $\widetilde{\mu}$ of the chemical potential and the single-particle ground state energy is determined by the equation

$$N = \sum_{j=0}^{\infty} \frac{j^2/2 + 3j/2 + 1}{\exp[(j\hbar\omega - \widetilde{\mu})/k_B T] - 1} \; , \tag{2}$$

where k_B denotes the Boltzmann constant, and T is the temperature. This sum can be evaluated approximately by introducing the density of states

$$\rho(E) \approx \frac{1}{2}\frac{E^2}{(\hbar\omega)^3} + \gamma\frac{E}{(\hbar\omega)^2} \tag{3}$$

with $\gamma = 3/2$, which follows immediately from eq. (1). Converting the sum into an integral and applying standard arguments, one then finds the condensation temperature [14–16]

$$T_C^{(3)} \approx T_0^{(3)} \left[1 - \frac{\gamma\,\zeta(2)}{3\,\zeta(3)^{2/3}} \cdot \frac{1}{N^{1/3}} \right] \; , \tag{4}$$

where

$$T_0^{(3)} = \frac{\hbar\omega}{k_B}\left(\frac{N}{\zeta(3)}\right)^{1/3} \tag{5}$$

denotes the condensation temperature pertaining to the large-N-limit [17,18]; $\zeta(z)$ is the Riemann zeta function. The lowering of $T_C^{(3)}$ with respect to $T_0^{(3)}$ is of order $N^{-1/3}$ and results from the enhancement of the density of states (3) above the leading "volume"-term: since there are more states available, the need to condense arises only at lower temperatures. It should be noted that, strictly speaking, there is no well-defined condensation temperature for a gas consisting of a finite number of particles. However, Fig. 1 clearly demonstrates that the onset of condensation in a three-dimensional harmonic trap becomes quite sharp already for particle numbers of the order of 10^5, and eq. (4) describes this onset very well.

In contrast to the textbook case of the free Bose gas, the heat capacity of the harmonically trapped Bose gas exhibits a steep drop at the condensation point. As shown in Fig. 2, this drop becomes a discontinuous jump by about 6.6 k_B per particle in the large-N-limit.

Remaining within the scope of the grand canonical ensemble, this analysis can be made formally more precise, and generalized to moderately anisotropic harmonic traps [19–22], which changes the value of γ. But can one rely on the grand canonical ensemble? Ziff, Uhlenbeck, and Kac have made a case that *the grand canonical ensemble does not represent any physical situation for the condensed ideal Bose gas*, and advocate that *its anomalous predictions should be ignored* [23]. These "anomalous predictions" are related to what we call the "grand canonical fluctuation catastrophe": within the grand canonical ensemble, the r.m.s.-fluctuations δN_0 of the ground state occupation number N_0 are given by [24,25]

$$(\delta N_0)^2 = N_0\,(N_0 + 1) \ , \tag{6}$$

implying that δN_0 is of the order of the total particle number N below the condensation point. However, if we consider an isolated Bose gas in a trap, all particles occupy the ground state at zero temperature, so that the true fluctuations of N_0 vanish. The grand canonical prediction for δN_0 thus differs drastically from the microcanonical one.

II. MICROCANONICAL APPROACH TO FLUCTUATIONS OF THE GROUND STATE OCCUPATION

Since the magnitude of the fluctuations δN_0 is related to the coherence properties of the condensate, it is of substantial interest to compute the true, i.e., microcanonical fluctuations for a trapped ideal Bose gas. We will study an isotropic harmonic trapping potential with oscillator frequency ω in d dimensions, and denote by n the number of excitation quanta for some preassigned value of the excitation energy E:

$$n = \frac{E}{\hbar\omega} \ . \tag{7}$$

The prime task now is to determine the number $\Omega^{(d)}(n|N)$ of microstates. Since there are generally many microstates where only a part of the N particles carries all n excitation quanta, leaving the other particles in the ground state, $\Omega^{(d)}(n|N)$ equals the number of possibilities for distributing the n quanta over *at most* N Bose particles. Then the difference

$\Omega^{(d)}(n|N_{\rm ex}) - \Omega^{(d)}(n|N_{\rm ex}-1)$ is the number of possibilities for distributing n quanta over *exactly* $N_{\rm ex}$ particles, with $N_{\rm ex}$ ranging from 1 to N, so that

$$p_{\rm ex}^{(d)}(N_{\rm ex}|n) = \frac{\Omega^{(d)}(n|N_{\rm ex}) - \Omega^{(d)}(n|N_{\rm ex}-1)}{\Omega^{(d)}(n|N)} \; , \qquad N_{\rm ex} = 1, 2, \ldots, N \; , \tag{8}$$

is the probability for finding $N_{\rm ex}$ out of N particles excited when the total excitation energy is $n \cdot \hbar\omega$. Since the remaining $N - N_{\rm ex}$ particles occupy the ground state, the first moment $\langle N_{\rm ex} \rangle$ of the distribution (8) yields die microcanonical expectation value of the ground state occupation number according to

$$\langle N_0 \rangle = N - \langle N_{\rm ex} \rangle \; ; \tag{9}$$

the corresponding fluctuations follow from

$$\left(\delta N_0 \right)^2 = \langle N_{\rm ex}^2 \rangle - \langle N_{\rm ex} \rangle^2 \; . \tag{10}$$

III. CONDENSATE FLUCTUATIONS IN A ONE-DIMENSIONAL OSCILLATOR POTENTIAL

The case $d = 1$ had been considered already in 1949 by Temperley [26], and shortly thereafter by Nanda [27]. It is of particular interest, since it can be treated in detail analytically with the tools furnished by the theory of the partion of integers: the number $\Omega^{(1)}(n|N)$ of microstates corresponds to the number of partitions of the integer n into at most N summands; the commutativity of the summands reflects the indistinguishability of the Bosons. We now have to distinguish two cases: if $n \le N$, then the fact that the number of particles is finite has no consequences for the enumeration of microstates, and we are dealing with *unrestricted* partitions of n. If $n > N$, the partitions of n are *restricted* by the requirement that the number of summands, corresponding to the number of excited particles, does not exceed N.

Let us first consider the case $n \le N$. We follow the usual convention and denote the number of unrestricted partitions of n as $p(n)$. Introducing the dimensionless inverse temperature $\xi = \hbar\omega/(k_B T)$, it is an elementary exercise to show that

$$\sum_{n=0}^{\infty} p(n) \, e^{-n\xi} = \prod_{j=1}^{\infty} \frac{1}{1 - \exp(-j\xi)} \equiv Z_{\infty}^{(1)}(\xi) \; , \tag{11}$$

which means that the generating function for $p(n)$ corresponds, physically speaking, to the canonical partition function of a fictituous system of infinitely many distinguishable harmonic oscillators with frequencies that are integer multiples of ω. Given this, it is natural to introduce the new variable $x = \exp(-\xi)$ and to extract the numbers $p(n)$ by inverting eq. (11) with the help of the saddle point approximation, after expressing ξ in terms of n. But we have to be careful: for temperatures that may physically be considered as “low”, but are still high compared to $\hbar\omega/k_B$, the variable $\exp(-\xi)$ is close to unity, so that we need to know the behaviour of $Z_{\infty}^{(1)}(\xi)$ in the vicinity of a singularity. However, there exists the remarkable identity (see eq. (1.42) in ref. [28])

$$Z_\infty^{(1)}(\xi) = \sqrt{\frac{\xi}{2\pi}}\,\exp\!\left(\frac{\zeta(2)}{\xi} - \frac{\xi}{24}\right)\, Z_\infty^{(1)}(4\pi^2/\xi) \tag{12}$$

that links the low-temperature behaviour of our fictituous oscillator system to its high-temperature dynamics. This allows us to derive the low-temperature approximation

$$\ln Z_\infty^{(1)}(\xi) \approx \frac{\zeta(2)}{\xi} + \frac{1}{2}\ln\xi - \frac{1}{2}\ln 2\pi \;, \tag{13}$$

which, in turn, yields the desired energy-temperature relation:

$$n \;=\; -\frac{\partial}{\partial\xi}\ln Z_\infty^{(1)}(\xi) \;\approx\; \frac{\zeta(2)}{\xi^2}\;. \tag{14}$$

Now we can compute the canonical entropy

$$S(\xi)/k_B = n\xi + \ln Z_\infty^{(1)}(\xi) \tag{15}$$

and apply "Bethe's theorem" [29] to get

$$p(n) \;\sim\; \frac{\exp\left[S(\xi(n))/k_B\right]}{\left(-2\pi\frac{\partial n}{\partial\xi}\right)^{1/2}} \;=\; \frac{1}{4\sqrt{3}n}\exp\left(\pi\sqrt{\frac{2}{3}n}\right)\;. \tag{16}$$

The right hand side is nothing but the celebrated Hardy–Ramanujan formula for the number of unrestricted partitions [28], so we have just sketched a physicist's solution to a number-theoretical problem. There is one detail that deserves particular attention: Bethe's theorem amounts to the inversion of the canonical partition sum (11) within the saddle point approximation, so that the dominant contributions to this sum are properly taken into account within the Gaussian approximation. As a result, the microcanonical entropy $\ln p(n)$ differs from the canonical entropy $S(\xi(n))/k_B$ by the saddle point correction $-\ln(-2\pi\partial n/\partial\xi)/2$. Such differences between thermodynamical quantities pertaining to different ensembles are characteristic for small systems. Expressed the other way round, the usually assumed equality of thermodynamical quantites in different ensembles holds to the extent that such saddle point corrections can be neglected.

Now we can turn to the case $n > N$, where the number of quanta exceeds the number of particles, and have to determine the number $\Omega^{(1)}(n|N) \equiv p_N(n)$ of restricted partitions of n. In principle, one can proceed as in the previous case, since there is the identity [30]

$$\sum_{n=0}^{\infty} p_N(n)\, e^{-n\xi} = \prod_{j=1}^{N}\frac{1}{1-\exp(-j\xi)} \equiv Z_N^{(1)}(\xi)\;\; : \tag{17}$$

The generating function for $p_N(n)$, that is, the canonical partition function for N ideal Bosons trapped by a one-dimensional harmonic potential, equals the canonical partition function of a system of N harmonic oscillators with frequencies ω, 2ω, …, $N\omega$. The corresponding asymptotic formula for $p_N(n)$, which turns out to be rather intricate, has been given by Auluck and Kothari [30]. However, a beautiful theorem due to Erdös and Lehner [31] helps to simplify the analysis: if, for some given n and x, the number N obeys

$$N = \frac{\sqrt{n}\ln n}{C} + x\sqrt{n} \qquad \text{with} \qquad C = \pi\sqrt{\frac{2}{3}}\ , \tag{18}$$

then

$$\lim_{n\to\infty} \frac{p_N(n)}{p(n)} \ = \ \exp\Big(-\frac{2}{C}e^{-\frac{1}{2}Cx}\Big)\ . \tag{19}$$

Hence, for given large values of n and N we *define* x according to

$$x = \frac{N}{\sqrt{n}} - \frac{\ln n}{C}\ , \tag{20}$$

and obtain the approximation

$$p_N(n) \approx p(n)\ \exp\Big(-\frac{2}{C}e^{-\frac{1}{2}Cx}\Big)\ . \tag{21}$$

In order to check this approximation, we first determine the numbers $p_N(n) = \Omega^{(1)}(n|N)$ by means of the saddle point inversion of eq. (17), evaluate the distributions (8), and compute the relative fluctuations $\delta N_0/N$ according to eq. (10). The results for $N = 10^3$, 10^4, and 10^5 are shown as full lines in Fig. 3. The corresponding data obtained with the help of the Erdös–Lehner approximation (21) are indicated by the dashed lines. Evidently, the approximation is quite good already for $N = 10^4$.

The merit of the approximation (21) lies in the fact that it allows us to derive an analytical expression for the microcanonical low-temperature ground state fluctuations of the ideal Bose gas trapped by a one-dimensional harmonic potential [32]:

$$\delta N_0 \approx \frac{\pi}{\sqrt{6}}\frac{k_BT}{\hbar\omega} \quad \text{for} \quad T \ll T_0^{(1)} \equiv \frac{\hbar\omega}{k_B}\frac{N}{\ln N}\ . \tag{22}$$

As already seen in Fig. 3, the low-temperature fluctuations vanish linearly with temperature. Most notably, they do *not* depend on the particle number N. This is immediately obvious for temperatures below the “restriction temperature” $T_R^{(1)}$ that is defined by the condition $n(T_R^{(1)}) = N$, since below $T_R^{(1)}$ the number of microstates becomes independent of N, so that “the condensate has no chance to know how many particles it consists of”. However, for $d = 1$ this restriction temperature is merely of the order of $N^{1/2}$, since $k_BT_R^{(1)} \approx \hbar\omega(N/\zeta(2))^{1/2}$, wheras the N-independence of δN_0 actually persists almost up to $k_BT_0^{(1)} = \hbar\omega(N/\ln N)$, where the occupation of the ground state becomes significant [32]. This N-independence of δN_0 appears to be characteristic for isolated condensates in general, as we shall indicate below.

IV. CONDENSATE FLUCTUATIONS IN A THREE-DIMENSIONAL TRAP

The analysis for the case $d = 3$ proceeds in close analogy to the case $d = 1$. The restriction temperature, determined from $n(T_R^{(3)}) = N$, now reads

$$T_R^{(3)} \approx 0.8\,N^{-1/12}\,T_0^{(3)}\ , \tag{23}$$

which means that $T_R^{(3)}$ is about one fourth of the condensation temperature for a gas consisting of 10^6 particles. Below $T_R^{(3)}$ the number of microstates, and hence all thermodynamical properties of the trapped gas, are independent of the particle number N. Accordingly, we write $\Omega^{(3)}(n|N) = \Omega^{(3)}(n)$ for n corresponding to temperatures T less than $T_R^{(3)}$, so that $\Omega^{(3)}(n)$ denotes the three-dimensional analogue of $p(n)$. The generating function for $\Omega^{(3)}(n)$ again equals the canonical partition function of a system of infinitely many, distinguishable harmonic oscillators [33]:

$$\sum_{n=0}^{\infty} \Omega^{(3)}(n)\, e^{-n\xi} = \prod_{j=1}^{\infty} \frac{1}{[1-\exp(-j\xi)]^{g_j}} \equiv Z_\infty^{(3)}(\xi) \; . \tag{24}$$

The saddle point inversion of this generating function is straightforward, but a bit tedious. The result has been given by Nanda in 1951 [34]:

$$\Omega^{(3)}(n) = \frac{\widetilde{n}^{-25/32}}{4\pi\left(3\zeta(4)\right)^{1/2}} \exp\left(4\zeta(4)\widetilde{n}^{3/4} + \frac{3}{2}\zeta(3)\widetilde{n}^{1/2} + \left[\zeta(2) - \frac{3}{8}\frac{\zeta(3)^2}{\zeta(4)}\right]\widetilde{n}^{1/4} + B\right) \tag{25}$$

with

$$\widetilde{n} = \frac{n}{3\zeta(4)}$$

and

$$B = \frac{\zeta(3)^3}{8\zeta(4)^2} - \frac{\zeta(2)\zeta(3)}{4\zeta(4)} + \frac{3}{2}\zeta'(-1) + \frac{1}{2}\zeta'(-2) \; .$$

This is one of the rare examples of a truly asymptotic formula for the number of microstates of a non-trivial Bose system.

For temperatures above $T_R^{(3)}$ the finiteness of the particle number restricts the number of microstates, and we have to compute $\Omega^{(3)}(n|N)$. It turns out that the logarithm of $\Omega^{(3)}(n)$ provides a quite good approximation to $\ln\left(\Omega^{(3)}(n|N)\right)$ even up to the condensation temperature [33], so that the entropy of the fictitious Boltzmannian oscillator system described by the partition function (24) approximates the entropy of the trapped Bose gas for all temperatures below the onset of condensation. This means that the distributions $p_{\rm ex}^{(3)}(N_{\rm ex}|n)$ introduced in eq. (8) must be well peaked; those microstates where the $n > N$ quanta are actually "spread out" over all N particles carry only a minor statistical weight below $T_C^{(3)}$. But still, this finding is of no help for computing the fluctuations δN_0, since this computation requires, according to eq. (8), to consider auxiliary systems consisting of $N_{\rm ex} \le N$ particles and to determine the numbers $\Omega^{(3)}(n|N_{\rm ex})$. The problem now is that an immediate analogue to the generating function (17) does not exist, that is, the canonical N-particle partition function $Z_N^{(3)}(\xi)$ is not known in closed form. However, there exists the recursion formula [35–37]

$$Z_N^{(d)}(\xi) = \frac{1}{N}\sum_{k=1}^{N} Z_1^{(d)}(k\xi) Z_{N-k}^{(d)}(\xi) \tag{26}$$

that can numerically be evaluated and inverted within the saddle point approximation, so that we can determine the distributions $p_{\rm ex}^{(3)}(N_{\rm ex}|n)$ at least numerically with high precision [33]. The results for $N = 1000$ are shown in Fig. 4: as anticipated, the distributions are well peaked, and moreover remarkably close to Gaussians.

From the expectation values of these distributions we obtain the microcanonical expectation values of the ground state occupation number, depicted as the full line in Fig. 5. The agreement with the corresponding grand canonical data (dashed) is stunning, but in view of the preceding Fig. 4 not unexpected: the predictions of the different ensembles will be the same as long as the most probable values of $N_{\rm ex}$ coincide with the expectation values, which is the case as long as the distributions $p_{\rm ex}^{(3)}(N_{\rm ex}|n)$ stay symmetrical. This, in turn, is guaranteed as long as the support of these distributions stays away from the border $N_{\rm ex}/N = 1$. Hence, there is a slight difference between the microcanonical and the grand canonical ground state fraction only in the immediate vicinity of the condensation point. For $d = 1$ this difference is much more pronounced, since $p_{\rm ex}^{(1)}(N_{\rm ex}|n)$ is considerably less well peaked [32] than its counterpart for $d = 3$.

Fig. 6 shows the microcanonical fluctuations of the ground state occupation number for $d = 3$ and $N = 200$, 500, and 1000. Remarkably, the low-temperature fluctuations for these three systems agree perfectly. More generally, if we compare two trapped Bose gases with different N under otherwise identical conditions, then their fluctuations δN_0 are nearly identical for temperatures below the lower of their condensation temperatures. This finding generalizes the formula (22). Again, the N-independence of δN_0 is immediately obvious for temperatures lower than the restriction temperature $T_R^{(3)}$, but it actually persists almost up to the condensation temperature, as a result of the well-peakedness of the distributions $p_{\rm ex}^{(3)}(N_{\rm ex}|n)$.

V. CONCLUDING REMARKS

We now come back to our initial question: Can we rely on the grand canonical ensemble for Bose–Einstein condensation in isolated traps? As long as we are only interested in the usual thermodynamical quantities, the answer is yes. Fig. 5 has shown that the grand canonical prediction for the ground state occupation number in a three-dimensional trap is almost indistinguishable from the microcanonical one even if the particle number N is as low as 1000. But in view of the fact that one-dimensional traps are not out of reach [16,38], it should be kept in mind that these differences are larger for $d = 1$ [32].

Another comparison between different ensembles is presented in Fig. 7, which depicts the grand canonical (long dashes), canonical (short dashes), and microcanonical (full line) specific heat capacity for $d = 3$ and $N = 1000$. Again, noticeable differences can be found only close to the onset of condensation.

But when it comes to quantum statistical properties, the grand canonical ensemble is unsound [23]. Adopting the microcanonical spirit and starting from the enumeration of microstates, we have computed the microcanonical low-temperature fluctuations δN_0 for harmonically trapped Bose gases. While the case $d = 1$ could be treated fully analytically, leading to eq. (22), the case $d = 3$ had to rely on the numerical evaluation of the recursion relation (26). If one could derive an analytical approximation to the numbers $\Omega^{(3)}(n|N)$ for $n > N$, with a quality similar to that of the Erdös–Lehner approximation (21) to $\Omega^{(1)}(n|N) \equiv$

$p_N(n)$, then the case $d = 3$ could be brought into the same textbook status that the case $d = 1$ already has by now.

It is also worthwhile to point out that very recently Navez *et al.* have proposed a new statistical ensemble, within which the ground state particles act as a reservoir, and exchange of particles with the excited-states subsystem without exchange of energy is possible [39]. The predictions for the fluctuations δN_0 obtained with the help of this so-called "Maxwell's Demon ensemble" agree very favourably with our strictly microcanonical results, so that this ensemble might constitute a valuable tool for further studies.

In closing, we reiterate the most important result of the present analysis: while the grand canonical ensemble predicts, for ideal Bose gases at low temperatures, ground state number fluctuations δN_0 of the order N, and while usual thermodynamical fluctuations scale as $\sqrt{N}$, the low-temperature fluctuations δN_0 for trapped, isolated ideal Bose condensates are independent of the total particle number N.

REFERENCES

[1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Science **269**, 198 (1995).

[2] K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).

[3] C.C. Bradley, C.A. Sackett, and R.G. Hulet, Phys. Rev. Lett. **78**, 985 (1997).

[4] M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.M. Kurn, D.S. Durfee, and W. Ketterle, Phys. Rev. Lett. **77**, 416 (1996).

[5] J.R. Ensher, D.S. Jin, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Phys. Rev. Lett. **77**, 4984 (1996).

[6] D.S. Jin, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Phys. Rev. Lett. **77**, 420 (1996).

[7] M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.M. Kurn, D.S. Durfee, C.G. Townsend, and W. Ketterle, Phys. Rev. Lett. **77**, 988 (1996).

[8] D.S. Jin, M.R. Matthews, J.R. Ensher, C.E. Wieman, and E.A. Cornell, Phys. Rev. Lett. **78**, 764 (1997).

[9] M.-O. Mewes, M.R. Andrews, D.M. Kurn, D.S. Durfee, C.G. Townsend, and W. Ketterle, Phys. Rev. Lett. **78**, 582 (1997).

[10] C.J. Myatt, E.A. Burt, R.W. Ghrist, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. **78**, 586 (1997).

[11] M.R. Andrews, C.G. Townsend, H.-J. Miesner, D.S. Durfee, D.M. Kurn, and W. Ketterle, Science **275**, 637 (1997).

[12] E.A. Burt, R.W. Ghrist, C.J. Myatt, M.J. Holland, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. **79**, 337 (1997).

[13] M.R. Andrews, D.M. Kurn, H.-J. Miesner, D.S. Durfee, C.G. Townsend, S. Inouye, and W. Ketterle, Phys. Rev. Lett. **79**, 553 (1997).

[14] S. Grossmann and M. Holthaus Phys. Lett. A **208**, 188 (1995).

[15] S. Grossmann and M. Holthaus Z. Naturforsch. **50 a**, 921 (1995).

[16] W. Ketterle and N.J. van Druten, Phys. Rev. A **54**, 656 (1996).

[17] S.R. de Groot, G.J. Hooyman, and C.A. ten Seldam, Proc. Roy. Soc. London A **203**, 266 (1950).

[18] V. Bagnato, D.E. Pritchard, and D. Kleppner, Phys. Rev. A **35**, 4354 (1987).

[19] K. Kirsten and D.J. Toms, Phys. Lett. A **222**, 148 (1996).

[20] K. Kirsten and D.J. Toms, Phys. Rev. A **54**, 4188 (1996).

[21] H. Haugerud, T. Haugset, and F. Ravndal, Phys. Lett. A **225**, 18 (1997).

[22] T. Haugset, H. Haugerud, and J.O. Andersen, Phys. Rev. A **55**, 2922 (1997).

[23] R.M. Ziff, G.E. Uhlenbeck, and M. Kac, Phys. Rep. **32**, 169 (1977). See p. 213.

[24] L.D. Landau and E.M. Lifshitz, *Statistical Physics* (Pergamon, London, 1959).

[25] R.K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1985).

[26] H.N.V. Temperley, Proc. Roy. Soc. London A **199**, 361 (1949).

[27] V.S. Nanda, Proc. Nat. Inst. Sci. (India) **19**, 681 (1953).

[28] G.H. Hardy and S. Ramanujan, Proc. Lond. Math. Soc. **17**, 75 (1918).

[29] H.A. Bethe, Rev. Mod. Phys. **9**, 69 (1937). See p. 81.

[30] F.C. Auluck and D.S. Kothari, Proc. Camb. Phil. Soc. **42**, 272 (1946).

[31] P. Erdös and J. Lehner, Duke Math. Jour. **8**, 335 (1941).

[32] S. Grossmann and M. Holthaus, Phys. Rev. E **54**, 3495 (1996).
[33] S. Grossmann and M. Holthaus, *Fluctuations of the particle number in a trapped Bose condensate*, preprint (1997).
[34] V.S. Nanda, Proc. Camb. Phil. Soc. **47**, 591 (1951).
[35] P. Borrmann and G. Franke, J. Chem. Phys. **98**, 2484 (1993).
[36] B. Eckhardt, *Eigenvalue statistics in quantum ideal gases*, preprint (1997).
[37] M. Wilkens and C. Weiss, *Universality classes and particle number fluctuations of trapped ideal Bose gases*, preprint (1997).
[38] N.J. van Druten and W. Ketterle, Phys. Rev. Lett. **79**, 549 (1997).
[39] P. Navez, D. Bitouk, M. Gajda, Z. Idziaszek, and K. Rzążewski, *The fourth statistical ensemble for the Bose–Einstein condensate*, preprint (1997).

FIGURES

FIG. 1. Grand canonical ground state fraction N_0/N for a three-dimensional isotropic harmonic trap, and gases consisting of 10^3, 10^5, and 10^7 ideal Bose particles. The larger the particle number, the sharper the onset of condensation, and the smaller the shift of the condensation temperature (4) with respect to $T_0 = T_0^{(3)}$. These data result from exact numerical calculations that do not invoke the continuum approximation.

FIG. 2. Grand canonical specific heat capacities for harmonically trapped ideal Bose gases ($d = 3$). The particle numbers are $N = 10^3$, 10^4, ..., 10^7. The higher the particle number, the steeper the drop at the onset of condensation.

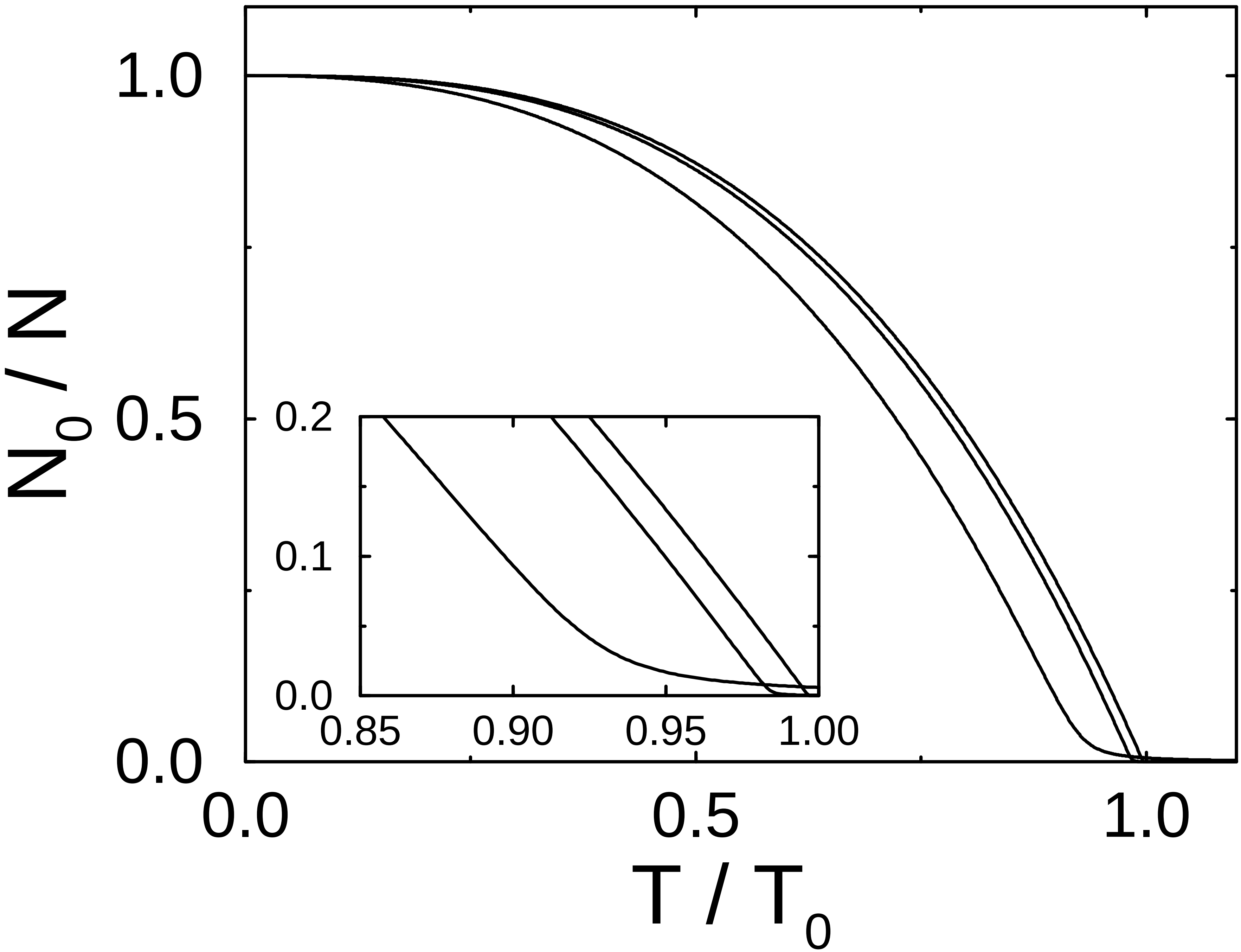
FIG. 3. Microcanonical fluctuations of the ground state occupation number for ideal Bose gases in a one-dimensional harmonic trap, for $N = 10^3$, 10^4, and 10^5 (top to bottom). Full lines: data obtained with the help of the saddle point inversion of eq. (17). Dashed: data resulting from the Erdös–Lehner formula (21). $T_0 = T_0^{(1)} \equiv \hbar\omega N/(k_B \ln N)$ marks the temperature below which the ground state occupation becomes significant [32]. The seeming N-dependence of the low-temperature fluctuations stems from the fact that they have been plotted versus the reduced temperature $T/T_0^{(1)}$, since $T_0^{(1)}$ depends on N.

FIG. 4. Microcanonical probability distributions $p_{\rm ex}^{(3)}(N_{\rm ex}|n)$ for finding $N_{\rm ex}$ out of $N = 1000$ isotropically trapped Bose particles excited when the total energy is $n \cdot \hbar\omega$. The temperatures $T/T_0^{(3)}$ corresponding to the Gaussian-like distributions range from 0.3 to 0.9 (left to right, in steps of 0.1); the temperature for the rightmost, monotonous distribution is $T = 0.95\,T_0^{(3)}$, which is higher than $T_C^{(3)} \approx 0.93\,T_0^{(3)}$, see eq. (4).

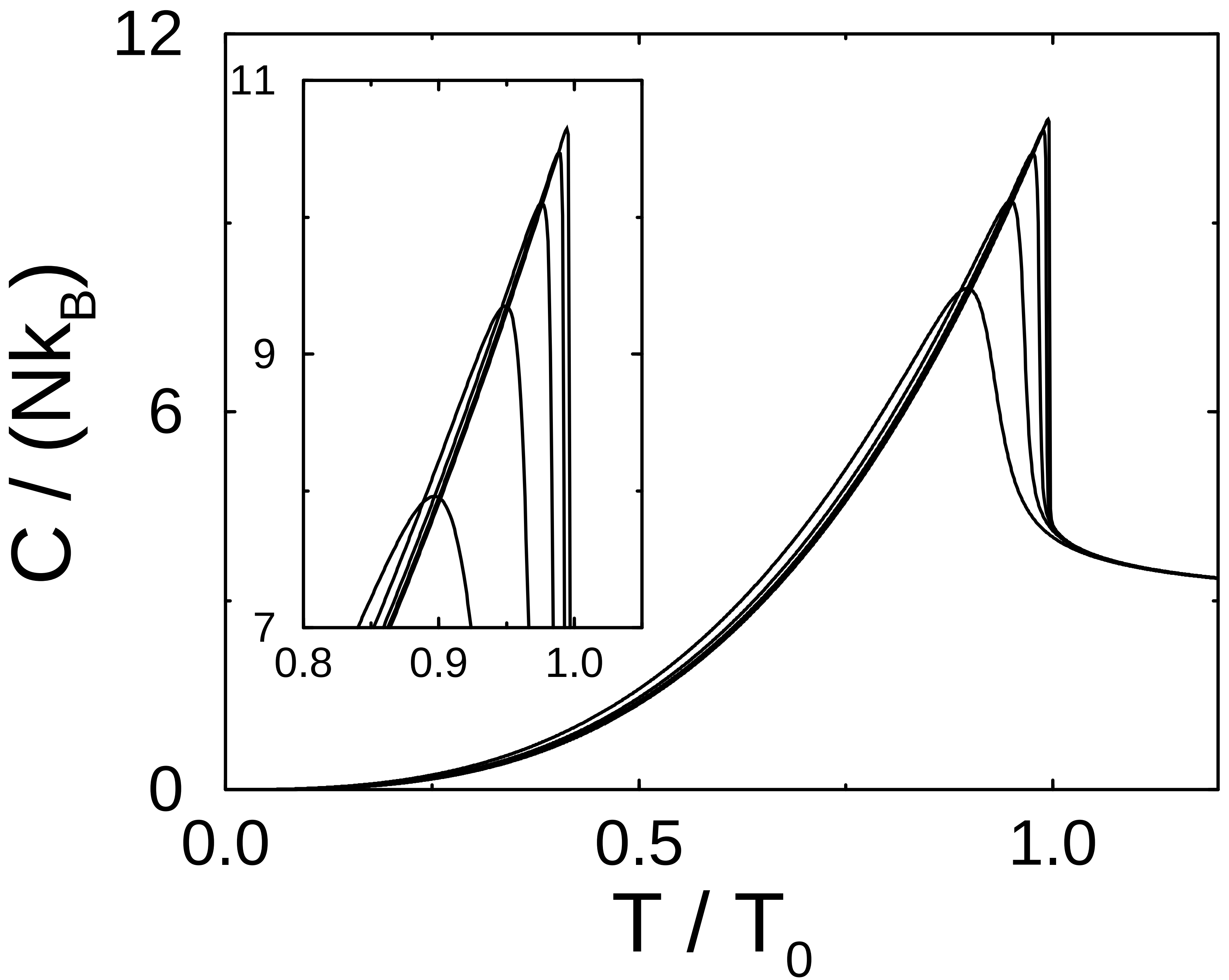
FIG. 5. Microcanonical ground state fraction (full line) for $N = 1000$ isotropically trapped Bose particles ($d = 3$) versus reduced temperature $T/T_0^{(3)}$, compared to the corresponding grand canonical data (dashed).

FIG. 6. Microcanonical fluctuations δN_0 versus temperature, for $d = 3$ and $N = 200$, 500, and 1000. The fluctuations are maximal close to the respective condensation points. Below the lowest of the condensation points, the fluctuations of the three systems agree perfectly, thus revealing the N-independence of δN_0.

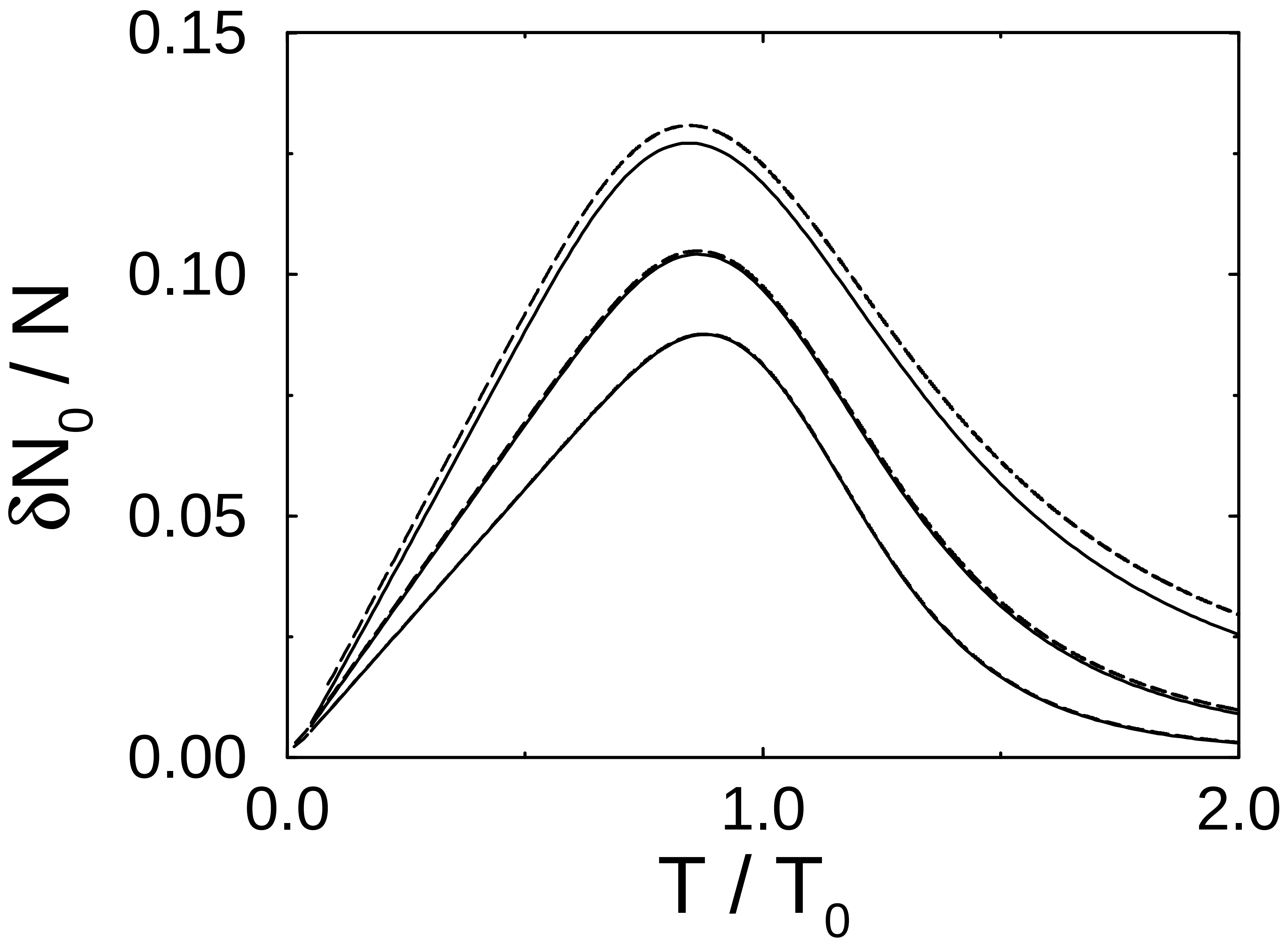
FIG. 7. Comparison of the grand canonical (long dashes), canonical (short dashes), and microcanonical (full line) specific heat capacity for $d = 3$ and $N = 1000$.

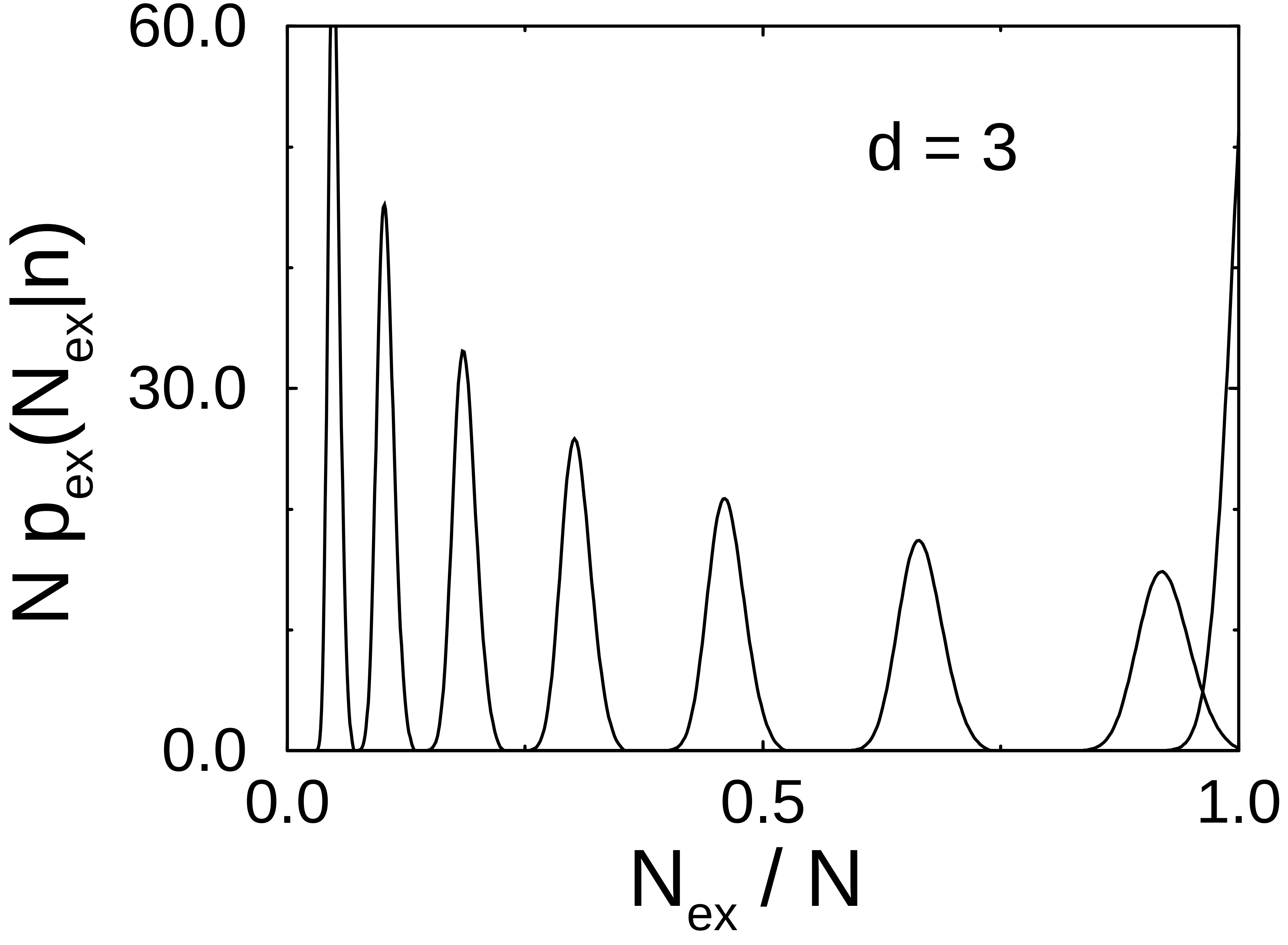


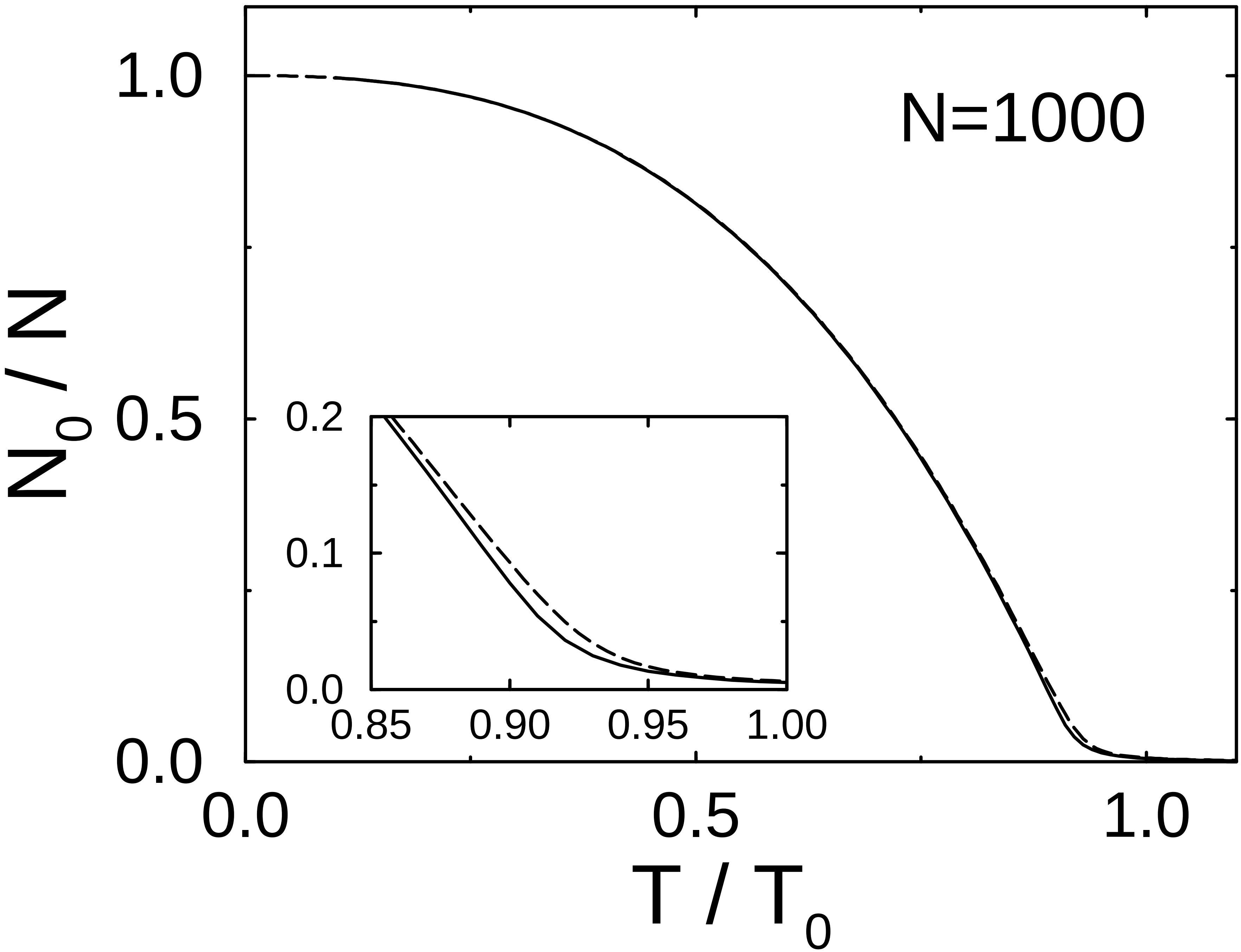
1.0
0.5
0.0
N_0 / N
0.0
0.5
1.0
T / T_0
0.2
0.1
0.0
0.85
0.90
0.95
1.00



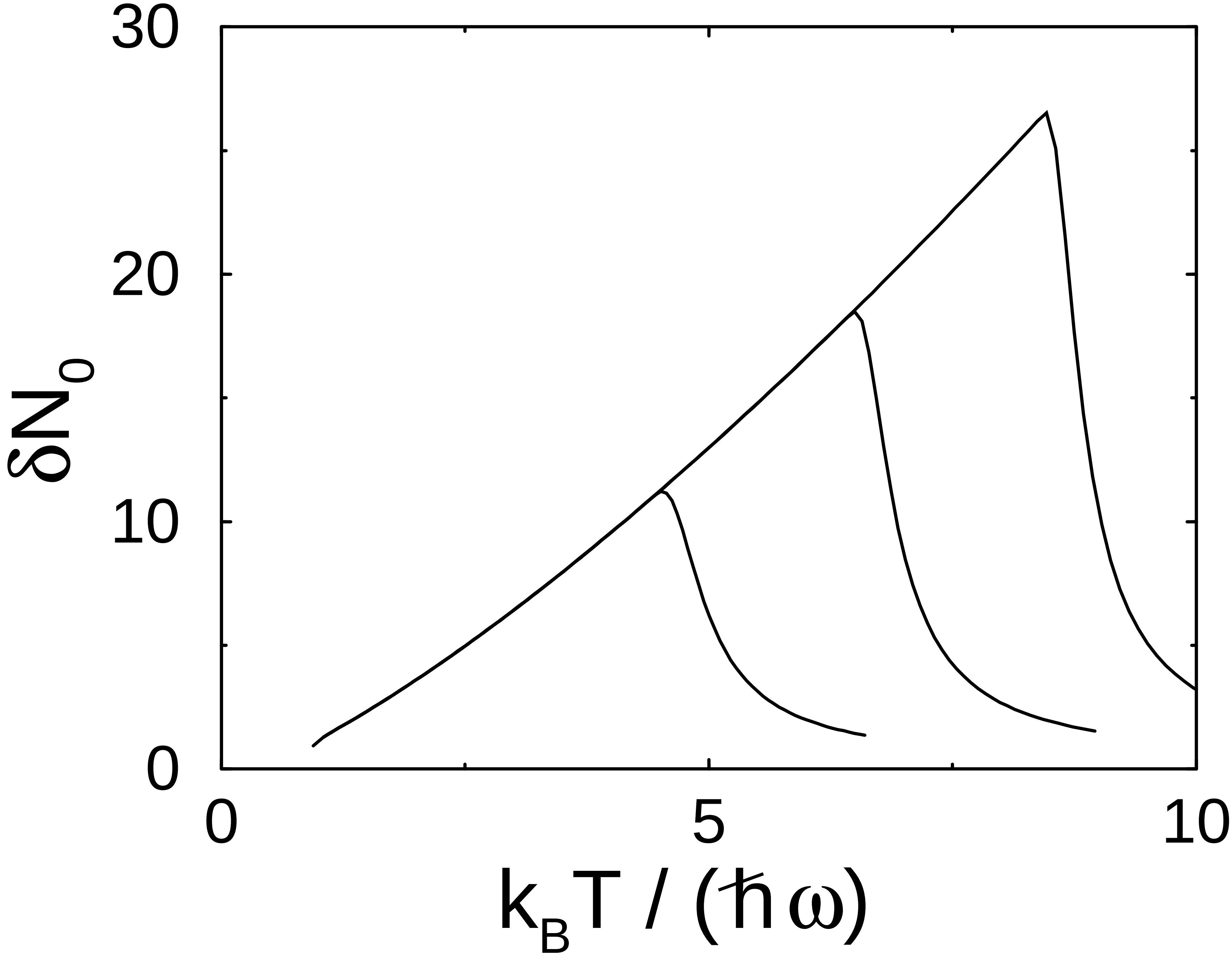
12
6
0
C / (Nk$_B$)
11
9
7
0.8
0.9
1.0
0.0
0.5
1.0
T / T$_0$


0.15
0.10
0.05
0.00
0.0
1.0
2.0
T / T0
δN0 / N

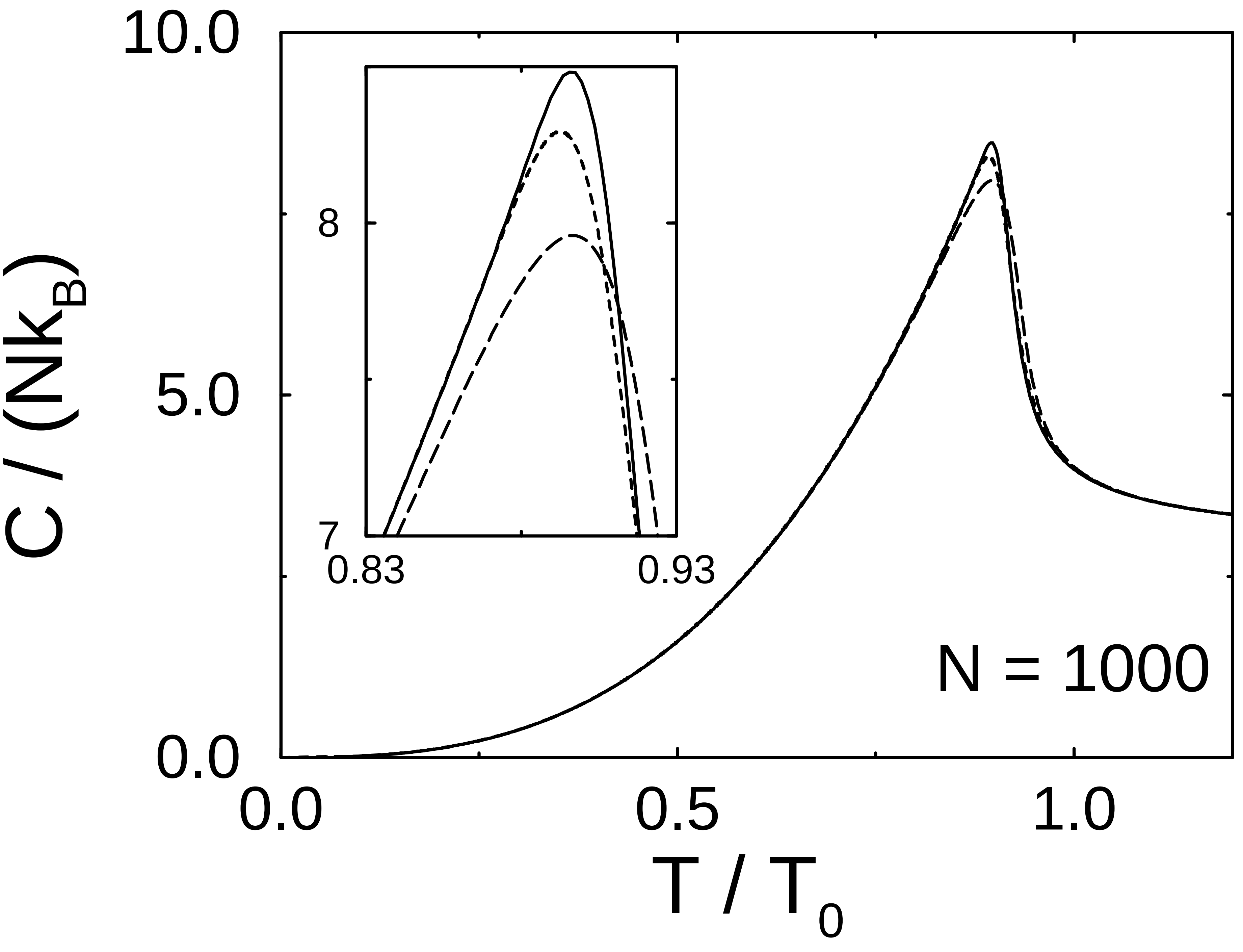

60.0
30.0
0.0
0.0
0.5
1.0
d = 3
N p_ex(N_ex|n)
N_ex / N



N=1000
N_0 / N
T / T_0
1.0
0.5
0.0
0.0
0.5
1.0
0.2
0.1
0.0
0.85
0.90
0.95
1.00



30
20
10
0
δN_0
0
5
10
$k_B T / (\hbar\omega)$


10.0
5.0
0.0
C / (Nk_B)
0.0
0.5
1.0
T / T_0
N = 1000
8
7
0.83
0.93