Condensed Matter Physics, 2012, Vol. 15, No 3, 33001: 1–9
DOI: 10.5488/CMP.15.33001
http://www.icmp.lviv.ua/journal

Asymptotic formulas for integer partitions within the approach of microcanonical ensemble

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Received May 5, 2012, in final form July 9, 2012

The problem of integer partitions is addressed using the microcanonical approach which is based on the analogy between this problem in the number theory and the calculation of microstates of a many-boson system. For ordinary (one-dimensional) partitions, the correction to the leading asymptotic is obtained. The estimate for the number of two-dimensional (plane) partitions coincides with known asymptotic results.

Key words: *integer partitions, plane partition, bosonic systems*

PACS: *05.30.Ch, 05.30.Jp*

1. Introduction

In this paper, we address the problem of integer partitions using the physical approach based on microcanonical treatment. Partitioning of integers is a problem in the number theory that originated in the works by Leibniz [1] and Euler [2]. A partition of a positive integer n is a way of writing n as a sum of positive integers, where the order of the summands is insignificant. The number of partitions $p(n)$ is called a *partition function* [3, Ch. 1] Further in this work we refer to $p(n)$ simply as *the number of partitions* to avoid confusion with the respective physical term.

To clarify the notion of partitions, let us consider the number 5. It can be represented as the following sums:

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

Therefore, one has for the number of partitions $p(5) = 7$.

Generalization for higher-dimensional partitions is made in the following fashion. In the D-dimensional case, an integer n is represented as a sum of positive integers $n_{i_1\dots i_D}$:

$$n = \sum_{i_1,\dots,i_D \geqslant 0} n_{i_1\dots i_D}\,, \tag{1.1}$$

where $n_{i_1\dots i_D} \geqslant n_{j_1\dots j_D}$ whenever $i_1 \leqslant j_1$, $i_2 \leqslant j_2$, $\dots$, $i_D \leqslant j_D$ [3, p. 179].

For instance, in the case of two-dimensional (plane) partitions of 3, one has the following possibilities:

$$\begin{array}{cccccc} 3 & 2+1 & 2 & 1+1+1 & 1+1 & 1 \\ & & + & & + & + \\ & & 1 & & 1 & 1 \\ & & & & & + \\ & & & & & 1 \end{array} \tag{1.2}$$

yielding the number of plane partitions $p_2(3) = 6$ [3, 4]. This can be easily shown by considering the numbers $n_{i_1 i_2}$ to be elements of square 1×1, 2×2, 3×3, and so on, matrices. To obtain the sum of matrix elements equal to 3 (provided that they are sorted in a non-ascending order in both rows and columns, according to the above definition), we have:

$$(3),\quad \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix},\quad \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix},\quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},\quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that zero elements are skipped when writing multidimensional partitions.

2. Physical analogy

There is a straight analogy between the number of partitions and the number of microstates of a many-boson system of harmonic oscillators. The approach based on such an analogy was suggested as early as in 1950 by Nanda [5], whose paper is now frequently overlooked, and was later utilized by a number of authors to solve related problems [6–8].

Let the system of one-dimensional harmonic oscillators with spectrum $\varepsilon_j = \hbar\omega j$ have an energy E:

$$E = \hbar\omega \sum_i j_i\,, \tag{2.1}$$

where j_i denotes the quantum number of the ith particle. The summation runs only over the excited states and thus the particles in the ground state (with zero energy) can be arbitrary in number. The set $\{j_1, j_2, \ldots\}$ corresponds to a certain microstate of the system. In the quantum case, the particles are indistinguishable, so the permutation of $\{j_1, j_2, \ldots\}$ does not lead to a new microstate. At this point, we can match the set $\{j_1, j_2, \ldots\}$ and a partition of the number $n = E/\hbar\omega$. Thus, the number of microstates $\Gamma(E)$ is equal to the number of partitions of n. In the set $\{j_1, j_2, \ldots\}$, any of the numbers j_i can be equal, so one must consider bosonic oscillators.

To make an asymptotic estimation of the number of microstates $\Gamma(E)$, one can use the well-known Hardy–Ramanujan formula [9] for integer partitions:

$$p^{\mathrm{HR}}(n) = \frac{1}{4\sqrt{3}\,n}\,\mathrm{e}^{\pi\sqrt{2/3}\,\sqrt{n}}, \tag{2.2}$$

therefore,

$$\Gamma(E) = \frac{1}{4\sqrt{3}\,E/\hbar\omega}\,\mathrm{e}^{\pi\sqrt{2/3}\,\sqrt{E/\hbar\omega}}. \tag{2.3}$$

In the next section, we will derive this expression from physical considerations and obtain the first correction to this asymptotic result. For simplicity, we further put the unit of energy $\hbar\omega = 1$.

The above considerations might be extended – with minor reservations – to higher-dimensional partitions, the two-dimensional (plane) ones being best studied. The respective analysis is presented in section 4.

3. Corrections to the leading asymptotics of $p(n)$

Partition function $Z(\beta)$ can be expressed as an integral by introducing the number of microstates $\Gamma(E)$:

$$Z(\beta) = \sum_j \mathrm{e}^{-\beta E_j} = \int_0^\infty \Gamma(E)\,\mathrm{e}^{-\beta E}\,\mathrm{d}E.$$

The above expression is nothing but the Laplace transform, so, by inverting it, we obtain:

$$\Gamma(E) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} Z(\beta)\,\mathrm{e}^{\beta E}\,\mathrm{d}\beta. \tag{3.1}$$

The entropy $S(\beta)$ is equal to

$$S(\beta) = \beta E + \ln Z(\beta). \tag{3.2}$$

Thus, we have

$$Z(\beta) = \mathrm{e}^{S(\beta)}\mathrm{e}^{-\beta E} \tag{3.3}$$

and

$$\Gamma(E) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathrm{e}^{S(\beta)}\mathrm{d}\beta. \tag{3.4}$$

Using the Taylor series for entropy in the vicinity of β_0:

$$S(\beta) \simeq S(\beta_0) + \frac{1}{2!}S''(\beta_0)(\beta-\beta_0)^2 + \frac{1}{3!}S'''(\beta_0)(\beta-\beta_0)^3, \tag{3.5}$$

where β_0 is the stationary point,

$$S'(\beta_0) = 0, \tag{3.6}$$

for the number of microstates, one obtains:

$$\Gamma(E) \simeq \frac{\mathrm{e}^{S(\beta_0)}}{2\pi \mathrm{i}} \int\limits_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \exp\left[\frac{1}{2!}S''(\beta_0)(\beta-\beta_0)^2 + \frac{1}{3!}S'''(\beta_0)(\beta-\beta_0)^3\right]\mathrm{d}\beta. \tag{3.7}$$

Using the replacement $\beta = \mathrm{i}x + \beta_0$, we get:

$$\begin{aligned} \Gamma(E) &\simeq \frac{\mathrm{e}^{S(\beta_0)}}{2\pi} \int\limits_{-\infty}^{\infty} \exp\left[-\frac{1}{2!}S''(\beta_0)x^2 - \frac{1}{3!}S'''(\beta_0)\mathrm{i}x^3\right]\mathrm{d}x \\ &= \frac{\mathrm{e}^{S(\beta_0)}}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2!}S''(\beta_0)x^2}\left\{\cos\left[\frac{S'''(\beta_0)}{3!}x^3\right] - \mathrm{i}\sin\left[\frac{S'''(\beta_0)}{3!}x^3\right]\right\}\mathrm{d}x \\ &= \frac{\mathrm{e}^{S(\beta_0)}}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2!}S''(\beta_0)x^2}\cos\left[\frac{S'''(\beta_0)}{3!}x^3\right]\mathrm{d}x. \end{aligned} \tag{3.8}$$

This integral can be expressed via the modified Bessel function of the second kind, namely:

$$\Gamma(E) \simeq \frac{\mathrm{e}^{S(\beta_0)}}{2\pi} \frac{2S''(\beta_0)}{\sqrt{3}\left|S'''(\beta_0)\right|} \exp\left\{\frac{[S''(\beta_0)]^3}{3[S'''(\beta_0)]^2}\right\} K_{1/3}\left(\frac{[S''(\beta_0)]^3}{3[S'''(\beta_0)]^2}\right). \tag{3.9}$$

To calculate the entropy $S(\beta)$, we write the partition function of the oscillator system as follows:

$$Z(\beta) = \prod_{j=1}^{\infty}\left(1-\mathrm{e}^{-\beta j}\right)^{-1}, \qquad \ln Z(\beta) = -\sum_{j=1}^{\infty}\ln\left(1-\mathrm{e}^{-\beta j}\right). \tag{3.10}$$

Using the Euler–Maclaurin formula to calculate the sum, one obtains [7]:

$$S(\beta) = \beta E + \ln Z(\beta) = \beta E + \frac{\pi^2}{6\beta} + \frac{1}{2}\ln\beta - \frac{1}{2}\ln(2\pi) + \ldots. \tag{3.11}$$

Limiting ourselves to the first two terms, the stationary point yields:

$$\beta_0 = \frac{\pi}{\sqrt{6E}}. \tag{3.12}$$

With the same accuracy, we have:

$$S''(\beta_0) = \frac{2\sqrt{6}}{\pi}E^{3/2}, \qquad S'''(\beta_0) = -\frac{36}{\pi^2}E^2. \tag{3.13}$$

Collecting the whole expression (3.9) together, for the number of microstates we obtain

$$\Gamma(E) = \frac{1}{18\sqrt[4]{6}\,E^{3/4}} \exp\left(\frac{28}{27}\pi\sqrt{\frac{2E}{3}}\right) K_{1/3}\left(\frac{1}{27}\pi\sqrt{\frac{2E}{3}}\right). \tag{3.14}$$

From this formula, there follows a correction to the main asymptotics of the number of partitions, as one substitutes E with n:

$$p(n) = \frac{1}{18\sqrt[4]{6}\,n^{3/4}} \exp\left(\frac{28}{27}\pi\sqrt{\frac{2n}{3}}\right) K_{1/3}\left(\frac{1}{27}\pi\sqrt{\frac{2n}{3}}\right). \tag{3.15}$$

Taking into account the asymptotic series expansion for large arguments [10]

$$K_\nu(z) \propto \sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-z}}{\sqrt{z}}\left[1+\mathscr{O}\left(\frac{1}{z}\right)\right] \tag{3.16}$$

we immediately arrive at the leading asymptotic behavior given by equation (2.2).

4. Asymptotic behavior of plane partitions

We consider the energy spectrum of a 2-dimensional system in the following form:

$$\varepsilon(j_1, j_2) = j_1 + j_2\,. \tag{4.1}$$

To obtain $\Gamma(E)$, we repeat the derivation presented in the previous section following [7], see also [11]. Thus, the entropy $S(\beta)$ is equal to

$$S(\beta) = \beta E + \ln Z(\beta), \tag{4.2}$$

where for energy spectrum (4.1), the partition function is

$$Z(\beta) = \prod_{\text{all the energies}} \left(1-\mathrm{e}^{-\beta\varepsilon}\right)^{-1} = \prod_{j_1=1}^{\infty}\prod_{j_2=1}^{\infty}\left[1-\mathrm{e}^{-\beta(j_1+j_2)}\right]^{-1}. \tag{4.3}$$

For the logarithm of the partition function $Z(\beta)$, one has:

$$\ln Z(\beta) = -\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\ln\left[1-\mathrm{e}^{-\beta(j_1+j_2)}\right] = -\sum_{j=1}^{\infty} g_j \ln\left[1-\mathrm{e}^{-\beta j}\right], \tag{4.4}$$

where the jth level degeneracy is equal to

$$g_j = j+1 \simeq j, \tag{4.5}$$

therefore,

$$\ln Z(\beta) = -\sum_{j=1}^{\infty} j \ln\left[1-\mathrm{e}^{-\beta j}\right]. \tag{4.6}$$

After applying the Euler–Maclaurin summation formula, the entropy can be expressed in such a form:

$$S(\beta) = \beta E + \ln Z(\beta) = \beta E + \frac{\zeta(3)}{\beta^2} + \frac{1}{12}\ln\beta - \frac{1}{6}\,. \tag{4.7}$$

The standard saddle-point method, for $\Gamma(E)$ from equation (3.1), yields:

$$\Gamma(E) = \frac{\exp[S(\beta_0)]}{\sqrt{2\pi S''(\beta_0)}}. \tag{4.8}$$

As in the previous section, for the stationary point β_0, so that $S'(\beta_0) = 0$, one obtains

$$\beta_0 = \left[\frac{2\zeta(3)}{E}\right]^{1/3} \tag{4.9}$$

and

$$S''(\beta_0) = \frac{3}{[2\zeta(3)]^{1/3}} E^{4/3}. \tag{4.10}$$

Thus, the number of microstates is

$$\Gamma(E) = \frac{1}{\sqrt{6\pi}} [2\zeta(3)]^{7/36} E^{-25/36} \exp\left\{\frac{3}{2}[2\zeta(3)]^{1/3} E^{2/3} - \frac{1}{6}\right\}. \tag{4.11}$$

Substituting energy E with integer n, we immediately obtain the result for plane partitions in the following way:

$$p_2(n) = \frac{1}{\sqrt{6\pi}} [2\zeta(3)]^{7/36} n^{-25/36} \exp\left\{\frac{3}{2}[2\zeta(3)]^{1/3} n^{2/3} - \frac{1}{6}\right\}. \tag{4.12}$$

Our estimation differs from the result of Wright [12]

$$p_2^{\mathrm{W}}(n) = \frac{1}{\sqrt{6\pi}} [2\zeta(3)]^{7/36} n^{-25/36} \exp\left\{\frac{3}{2}[2\zeta(3)]^{1/3} n^{2/3} + c\right\} \tag{4.13}$$

by a constant factor:

$$c = \zeta'(-1) = -0.165421\ldots \quad \text{versus} \quad -\frac{1}{6} = -0.166666\ldots.$$

Note that Wright's asymptotics was later confirmed by Nanda [5] with a much more sophisticated analysis of the oscillator system as compared to our approach.

5. Results and discussion

We performed calculations for the number of ordinary partitions $p(n)$ and plane partitions $p_2(n)$ using the expressions (3.15) and (4.12) obtained in this work.

The comparison with exact values and the leading asymptotics by Hardy and Ramanujan are given in table 1 and in figure 1. In the figure, the relative errors are plotted. For $n = 32$ and $n > 33$, our correction becomes negative. For $n > 20$, the error is less than one per cent, and it never exceeds seven per cent except for $n = 1$, where $p(n) = 1$, and our formula yields 2, and $n = 5$ with our value of 8 versus $p(n) = 5$, which leads to about 15 per cent error. Beyond this domain, the maximum relative deviation of 0.0083285 is found for $n = 186$.

The plane partitions are demonstrated in table 2 and in figure 2. Our result for plane partitions provides a better estimation compared to that of Wright up to $n = 7573$. At $n = 2679$, expression (4.12) starts to underestimate the number of plane partitions while (4.13) asymptotically approaches the real values of $p_2(n)$ from above for all n. Interestingly, the approach to the calculation of plain partitions used in this work which takes into account the level degeneracy provides a much better estimate compared to an earlier suggested [15] direct treatment of multidimensional oscillators which failed to produce a correct pre-exponential behavior of the power of n.

To summarize, the expressions for the number of integer partitions were obtained from the analogy between this number-theoretical problem and the physical problem of calculating the states in a many-boson system within the microcanonical approach. The correction to the main asymptotics for ordinary (one-dimensional) partitions is shown to give a good estimate even for small numbers.

Table 1. Number of integer partitions. Real data $p(n)$ are calculated by `wxMaxima 0.8.7`. Our result (3.15) is denoted as $p^{\text{our}}(n)$.

n	$p(n)$	$p^{\text{HR}}(n)$	$p^{\text{our}}(n)$
1	1	2	2
2	2	3	2
3	3	4	3
4	5	6	5
5	7	9	8
6	11	13	11
7	15	18	16
8	22	26	23
9	30	35	31
10	42	48	43
20	627	692	631
30	5 604	6 080	5 607
40	37 338	40 080	37 244
50	204 226	217 590	203 334
60	966 467	1 024 004	961 084
70	4 087 968	4 312 670	4 061 899
80	15 796 476	16 606 782	15 686 810
90	56 634 173	59 367 760	56 218 131
100	190 569 292	199 280 893	189 113 660
150	40 853 235 313	42 369 336 269	40 515 857 434
200	3 972 999 029 388	4 100 251 432 188	3 939 941 762 556
300	9 253 082 936 723 602	9 494 094 811 675 004	9 178 612 996 544 433
400	6 727 090 051 741 041 926	6 878 471 626 940 064 454	6 675 403 544 180 355 365
500	2 300 165 032 574 323 995 027	2 346 386 625 611 060 168 255	2 283 287 889 193 750 426 745
600	458 004 788 008 144 308 553 622	466 396 419 561 000 383 265 349	454 786 235 887 791 524 386 094
700	60 378 285 202 834 474 611 028 659	61 401 534 136 286 099 837 866 892	59 970 590 732 043 189 238 732 265
800	5 733 052 172 321 422 504 456 911 979	5 823 869 045 997 298 219 672 106 135	5 695 741 931 526 069 206 199 862 701
900	415 873 681 190 459 054 784 114 365 430	422 080 911 932 431 823 414 681 187 746	413 258 060 326 350 156 090 055 206 821
1 000	24 061 467 864 032 622 473 692 149 727 991	24 401 996 316 802 476 288 263 414 943 062	23 914 862 944 527 589 687 173 572 605 174
10 000	3.61673×10^{106}	3.63281×10^{106}	3.60761×10^{106}
100 000	2.74935×10^{346}	2.75321×10^{346}	2.74695×10^{346}
200 000	1.14215×10^{492}	1.14328×10^{492}	1.14144×10^{492}
500 000	1.52473×10^{781}	1.52568×10^{781}	1.52412×10^{781}
1 000 000	1.47168×10^{1107}	1.47234×10^{1107}	1.47127×10^{1107}

Table 2. Number of plane integer partitions, real data $p_2(n)$ are presented according to [13, 14]. Our result is denoted as $p_2^{\text{our}}(n)$.

n	$p_2(n)$	$p_2^{\text{W}}(n)$	$p_2^{\text{our}}(n)$
1	1	2	2
2	3	3	3
3	6	7	7
4	13	14	14
5	24	27	27
6	48	51	51
7	86	94	93
8	160	169	169
9	282	300	300
10	500	526	525
15	6 879	7 174	7 165
20	75 278	77 828	77 731
25	696 033	716 466	715 574
30	5 668 963	5 814 929	5 807 691
40	281 846 923	287 805 195	287 446 950
50	10 499 640 707	10 690 144 561	10 676 838 029
60	314 689 799 781	319 730 938 404	319 332 954 078
70	7 937 771 067 795	8 052 251 247 267	8 042 228 226 656
80	173 781 688 194 937	176 070 453 255 058	175 851 289 976 638
90	3 376 508 618 954 817	3 417 560 875 691 699	3 413 306 874 910 366
100	59 206 066 030 052 023	59 876 276 156 314 650	59 801 745 303 809 873
125	52 658 376 905 566 496 345	53 170 908 672 841 062 211	53 104 724 310 582 212 659
150	30 669 139 297 980 503 425 545	30 933 055 614 865 435 615 600	30 894 551 767 372 734 822 993
175	12 797 410 813 969 092 203 145 839	12 896 659 914 533 283 896 907 383	12 880 606 827 741 697 065 333 190
200	4 066 263 490 068 623 016 919 082 185	4 095 085 410 386 804 831 530 514 486	4 089 988 062 550 422 858 471 571 831
1 000	3.54259×10^{84}	3.55112×10^{84}	3.54670×10^{84}
10 000	4.50750×10^{401}	4.50983×10^{401}	4.50422×10^{401}
100 000	1.11796×10^{1876}	1.11808×10^{1876}	1.11669×10^{1876}

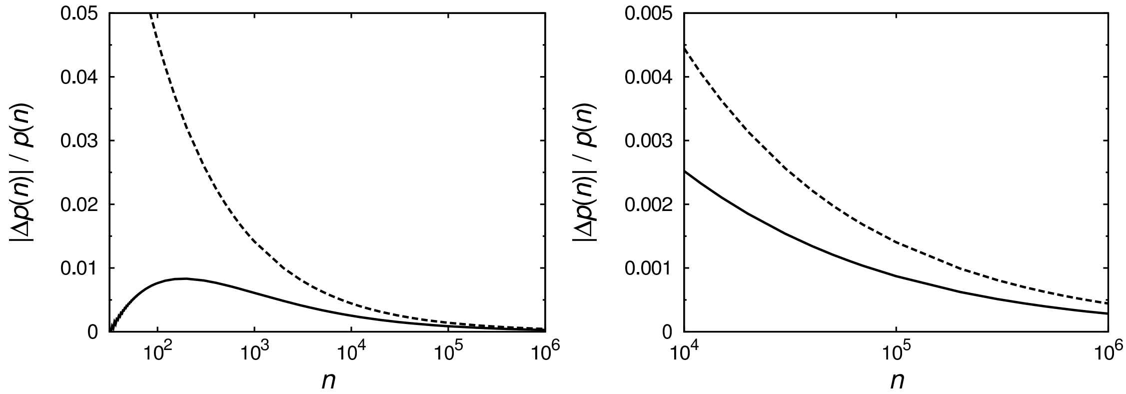

Figure 1. Comparison of relative errors for the estimations of the number of integer partitions from the real values (calculated by `wxMaxima 0.8.7`). Solid curve – our result (3.15), dashed curve – the leading asymptotics provided by the Hardy–Ramanujan formula (2.2). Large n domain in an enlarged view is presented on the right.

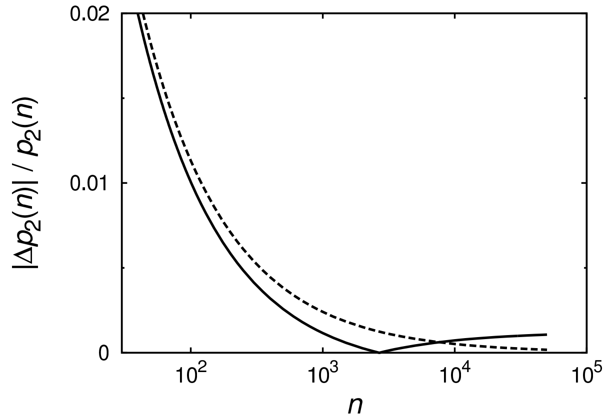

Figure 2. Comparison of relative errors for the estimations of the number of plane partitions from the real values $p_2(n)$ [13, 14]. Solid curve – our result (4.12), dashed curve – Wright's formula (4.13).

Acknowledgements

We thank Yuri Krynytskyi for discussions on some issues raised in this paper. The work was partly supported by grant ФФ–110Ф (No. 0112U001275) from the Ministry of education, science, youth, and sports of Ukraine.

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Асимптотичні формули для розбиттів цілих чисел у підході мікроканонічного ансамблю

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Розглянуто задачу про розбиття цілих чисел у межах мікроканонічного підходу, який ґрунтується на аналогії між цією задачею з теорії чисел і обчисленням кількості мікростанів багатобозонної системи. Для звичайних (одновимірних) розбиттів отримано поправку до головної асимптотики. Оцінка кількості двовимірних (плоских) розбиттів добре узгоджується з відомими асимптотичними результатами.

Ключові слова: *розбиття цілих чисел, плоскі розбиття, бозонні системи*