Theoretical and Mathematical Physics, **181**(2): 1428–1434 (2014)

ENUMERATION OF PLANE PARTITIONS WITH A RESTRICTED NUMBER OF PARTS

© **A. A. Rovenchak***

We use the quantum statistical approach to estimate the number of restricted plane partitions of an integer n with the number of parts not exceeding some finite N. We use the analogy between this number theory problem and the enumeration of microstates of the ideal two-dimensional Bose gas. The numbers of restricted plane partitions calculated with the conjectured expression agree well with the exact values for n from 10 to 20.

Keywords: Bose gas, plane partition, restricted partition

1. Introduction

The problem of integer partitions, already originating in works of Leibniz and Euler, has found numerous applications not only in mathematics but also in different domains of physics. Domains of mathematics where it is used, for example, include combinatorics and probability theory, while in physics, this problem is related to the theory of crystals, percolation theory, and also quantum statistics (see, e.g., [1] and the references therein).

So-called two-dimensional or plane partitions are a special type of integer partitions. A plane partition of a positive integer number n is a two-dimensional array of nonnegative integers n_{ij} satisfying a nonincrease condition across rows and columns such that

$$n = \sum_{i,j>0} n_{ij}, \quad n_{i_1 j_1} \ge n_{i_2 j_2} \quad \text{for } i_1 \le i_2, \quad j_1 \le j_2$$

(see [2]). For instance, all 13 two-dimensional partitions of the number 4 are [3]

$$4,\quad 3\,1,\quad \begin{matrix}3\\1\end{matrix},\quad 2\,2,\quad \begin{matrix}2\\2\end{matrix},\quad 2\,1\,1,\quad \begin{matrix}2\,1\\1\end{matrix},\quad \begin{matrix}2\\1\\1\end{matrix},\quad 1\,1\,1\,1,\quad \begin{matrix}1\,1\,1\\1\end{matrix},\quad \begin{matrix}1\,1\\1\,1\end{matrix},\quad \begin{matrix}1\,1\\1\\1\end{matrix},\quad \begin{matrix}1\\1\\1\\1\end{matrix}.$$

Zero elements are traditionally suppressed when writing partitions; the remaining nonzero elements are called parts. The number of different plane partitions of n is further denoted by $p^{2\mathrm{D}}(n)$; $p^{2\mathrm{D}}(4) = 13$ in the above example. This quantity is traditionally called the "partition function" in mathematics. To avoid ambiguities here, we reserve the term "partition function" for the *Zustandsumme* in statistical physics and call $p^{2\mathrm{D}}(n)$ the *number of partitions.*

*Department for Theoretical Physics, Ivan Franko National University of Lviv, Lviv, Ukraine, e-mail: andrij.rovenchak@gmail.com.

Prepared from an English manuscript submitted by the author; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 181, No. 2, pp. 349–357, November, 2014. Original article submitted January 19, 2014; revised April 22, 2014.

As with simple one-dimensional or linear partitions [4]–[7], the problem of enumerating plane partitions can be related to the problem of counting the number of microstates in a system of two-dimensional quantum harmonic oscillators obeying the Bose–Einstein statistics [5], [8].

Different types of restrictions can be imposed on partitions. We can require the parts to be either odd or even numbers, limit the magnitude or the number of parts, and so on [2]. With respect to quantum ensembles, this in particular corresponds to studies of fractional statistics or effects of a finite number of particles [6], [9], [10]. With plane partitions, it is also possible to impose different shapes, to limit the number of rows and columns, and so on [2], [11], [12]. Curiously enough, it seems that the problem of enumerating plane partitions with the sole restriction on the number of parts (considered from the standpoint of asymptotic behavior) has not been properly reflected in the literature. Our aim here is to partially fill this gap.

This paper is organized as follows. Section 2 contains a derivation of the sought expressions for a finite N-particle system in the general D-dimensional case. These expressions are then applied to the one-dimensional problem, whence we derive the known behavior of ordinary (linear) restricted partitions in Sec. 3. We consider restricted plane partitions in Sec. 4. A short discussion concludes the paper.

2. General results for a finite system of N particles

The partition function Z_N of a finite system of N bosonic harmonic oscillators satisfies the recurrence relation [6], [13]

$$Z_N(x) = \frac{1}{N}\sum_{k=1}^{N} B_k(x) Z_{N-k}(x), \qquad Z_0(x) \equiv 1, \tag{1}$$

where $x = e^{-\beta\hbar\omega}$, β is the inverse temperature, and ω is the oscillator frequency. In D dimensions,

$$B_k(x) = \frac{1}{(1-x^k)^D}.$$

A closed-form expression for Z_N exists only in the one-dimensional case:

$$Z_N^{1\mathrm{D}}(x) = \prod_{k=1}^{N} \frac{1}{1-x^k}. \tag{2}$$

In what follows, we use this result to verify the proposed method.

To solve Eq. (1), we can apply the integral transformation. To avoid complications connected with passing from summation to integration, it seems our best choice is to use a discrete transformation.

The Z-transform of a function $f(N)$ is defined as [14]

$$\mathfrak{Z}[f(N)] = \sum_{n=0}^{\infty} f(n) s^{-n} = \tilde{f}(s).$$

It is a discrete analogue of the Laplace transform. The two properties of the Z-transform are required for solving Eq. (1):

$$\mathfrak{Z}[N f(N)] = -s\frac{d\tilde{f}(s)}{ds}, \qquad \mathfrak{Z}[f(N) * g(N)] = \tilde{f}(s)\tilde{g}(s),$$

where the convolution is defined as

$$f(N) * g(N) = \sum_{n=0}^{N} f(n) g(N-n).$$

We rewrite Eq. (1) in the form immediately suitable for using the Z-transform:

$$NZ_N(x) = \sum_{k=0}^{N} B_k(x) Z_{N-k}(x), \qquad B_0(x) \stackrel{\text{def}}{=} 0.$$

It seems more convenient to consider the correction to the partition function of an infinite system $Z_\infty(x)$:

$$Z_N(x) = Z_\infty(x) y_N(x), \tag{3}$$

where the function $y_N(x)$ has the obvious limit behavior

$$\lim_{N\to\infty} y_N(x) = 1. \tag{4}$$

For the transform of this correction, we easily obtain

$$-s\frac{d\tilde{y}(s|x)}{ds} = \widetilde{B}(s|x)\tilde{y}(s|x) \tag{5}$$

or

$$\tilde{y}(s|x) = C\exp\left\{-\int^{s} \frac{\widetilde{B}(s'|x)}{s'}\, ds'\right\},$$

where the integration constant C can be found from Eq. (4).

3. Testing the approach in one dimension

We first verify the method in the one-dimensional case, where all results are well known [6], [7], [15]. The summation in the transform of $B_N(x)$ is easily done in the first order of x:

$$\widetilde{B}^{\mathrm{1D}}(s|x) = \sum_{k=1}^{\infty} \frac{s^{-k}}{1-x^k} \simeq \sum_{k=1}^{\infty} s^{-k}(1+x^k) = \frac{s-2x+sx}{(s-1)(s-x)},$$

which gives

$$\tilde{y}^{\mathrm{1D}}(s|x) = C\frac{s^2}{(s-1)(s-x)},$$

and inverting the transformation, we therefore obtain

$$y_N^{\mathrm{1D}}(x) = \mathfrak{Z}^{-1}\left[C\frac{s^2}{(s-1)(s-x)}\right] = C\frac{x^{N+1}-1}{x-1}. \tag{6}$$

From Eq. (4), we find the integration constant $C = 1-x$ and finally in the leading order

$$y_N^{\mathrm{1D}}(x) = 1 - x^{N+1}.$$

This result correctly reproduces exact expression (2). Indeed,

$$Z_\infty^{\mathrm{1D}}(x) = \prod_{k=1}^{N} \frac{1}{1-x^k} \prod_{k=N+1}^{\infty} \frac{1}{1-x^k},$$

hence

$$y_N^{1\mathrm{D}}(x) = \prod_{k=N+1}^{\infty} (1 - x^k) = \exp \sum_{k=N+1}^{\infty} \log(1 - x^k). \tag{7}$$

As before, in the leading order, taking into account that $x < 1$ and N is large, we obtain

$$y_N^{1\mathrm{D}}(x) = e^{-x^{N+1}} = 1 - x^{N+1} \pm \dots . \tag{8}$$

Some clarifications are required here. The number of (one-dimensional or linear) partitions $p^{1\mathrm{D}}(n) \equiv p(n)$ of an integer n is equal to the number of microstates $\Gamma(E)$ of the system with the energy $E = \hbar\omega n$. The function $\Gamma(E)$ is related to $Z(\beta)$ via the inverse Laplace transformation, which we can evaluate using the steepest-descent method [7]:

$$\Gamma(E) = \frac{e^{S(\beta_0)}}{\sqrt{2\pi S''(\beta_0)}}, \tag{9}$$

where the entropy is $S(\beta) = \beta E + \log Z(\beta)$ and the stationary point β_0 is defined by $S'(\beta_0) = 0$.

Considering a finite number of particles N (or, equivalently, a finite number of parts for the partitions), from Eq. (3), we can see that the expression for $y_N(x)$ should directly enter the formula for the number of restricted partitions. Indeed, for a finite N, the entropy equals $S_N = \beta E + \log Z_\infty + \log y_N$, which by virtue of Eq. (9) yields

$$p_N(n) = p(n) y_N(e^{-\beta_0}),$$

where the stationary point is $\beta_0 = \pi/\sqrt{6n}$ [7], [8]. We note that because n is large, β_0 is small, and the argument $x = e^{-\beta_0}$ is close to unity, although still $x < 1$. The summation in (7) therefore requires a more careful approach, namely,

$$\sum_{k=N+1}^{\infty} \log(1 - x^k) \simeq - \sum_{k=N+1}^{\infty} x^k = -\frac{x^{N+1}}{1 - x}.$$

With $N \gg 1$, this provides a modification of (8):

$$y_N^{1\mathrm{D}}(x) = \exp\biggl(-\frac{x^N}{1 - x}\biggr) \simeq \exp\biggl(-\frac{x^N}{\beta_0}\biggr).$$

Therefore, the leading correction in the number of restricted partitions is [4], [7]

$$p_N(n) = p(n) \exp\biggl\{-\frac{\sqrt{6n}}{\pi} e^{-\pi N/\sqrt{6n}}\biggr\}, \tag{10}$$

reproducing the classical result of Erdős and Lehner [16] about the asymptotic behavior of the number of partitions of n into at most N parts.

Obtaining such an expression (10) directly from Eq. (6) would be somewhat speculative because the Z-transformed function $\widetilde{B}^{1\mathrm{D}}(s)$ is derived in the limit of small x. We can solve the equation for $\tilde{y}^{1\mathrm{D}}(s)$ in a closed form in the limit as $x \to 1$, but the inversion $\mathfrak{Z}^{-1}[\tilde{y}^{1\mathrm{D}}(s)]$ cannot be done analytically in this case. We can therefore use the relation between the dependences on N and on x for $y_N^{1\mathrm{D}}(x)$ obtained using the two different approaches and assume that a similar transition also holds in higher space dimensions. As is shown below, this assumption leads to a quite good agreement between actual and calculated numbers of restricted plane partitions.

4. Results for plane partitions

In two dimensions, $B_k^{\rm 2D}(x)$ equals

$$B_k^{\rm 2D}(x) = \frac{1}{(1-x^k)^2},$$

which leads to the approximate expression

$$\widetilde{B}^{\rm 2D}(s|x) \simeq \sum_{k=1}^{\infty} s^{-k}(1+2x^k) = \frac{s-3x+2sx}{(s-1)(s-x)}.$$

The solution of Eq. (5) is

$$\tilde{y}^{\rm 2D}(s|x) = C\,\frac{s^3}{(s-1)(s-x)^2}.$$

Its inverse Z-transform is

$$y_N^{\rm 2D}(x) = \mathfrak{Z}^{-1}[\tilde{y}^{\rm 2D}(s|x)] = C\,\frac{(N+1)x^{N+2}-(N+2)x^{N+1}+1}{(x-1)^2},$$

where $C=(x-1)^2$; for large N and small x, it becomes

$$y_N^{\rm 2D}(x) = 1 - Nx^N. \tag{11}$$

A similar expression can be obtained for a system of N isotropic two-dimensional oscillators if the partition function is written in the form

$$\log Z_N^{\rm 2D}(x) = -\sum_{k=1}^{N} k\log(1-x^k), \tag{12}$$

where the k-fold degeneracy of the kth level is taken into account. If we replace the upper summation limit with infinity, then we obtain MacMahon's generating function for plane partitions [17]:

$$\sum_{n=0}^{\infty} p^{\rm 2D}(n)x^n = \prod_{n=1}^{\infty}\frac{1}{(1-x^n)^n}.$$

Comparing Eqs. (11) and (12) and acting by analogy with the one-dimensional case, we consider the limit $\beta\to 0$. For $y_N^{\rm 2D}(x)$, we can obtain

$$y_N^{\rm 2D}(x) = \exp\sum_{k=N+1}^{\infty} k\log(1-x^k) \simeq$$

$$\simeq \exp\biggl(-\sum_{k=N+1}^{\infty} kx^k\biggr) = \exp\biggl(-\frac{x^{N+1}[N(1-x)+1]}{(1-x)^2}\biggr).$$

For $N\beta_0 \gg 1$, we thus obtain the asymptotic formula

$$y_N^{\rm 2D}(x) = \exp\biggl(-\frac{Nx^N}{1-x}\biggr) \simeq \exp\biggl(-\frac{Nx^N}{\beta_0}\biggr)$$

with the stationary point

$$\beta_0 = \left(\frac{2\zeta(3)}{n}\right)^{1/3},$$

where $\zeta(x)$ denotes the Riemann zeta function [8].

Therefore, we can conjecture the asymptotic behavior

$$p_N^{2\mathrm{D}}(n) = p^{2\mathrm{D}}(n)\exp\biggl\{-\frac{Nn^{1/3}}{[2\zeta(3)]^{1/3}}e^{-N[2\zeta(3)/n]^{1/3}}\biggr\} \tag{13}$$

for the number of restricted plane partitions. This formula is our main result. It estimates the number of plane partitions of n into at most N parts.

The conditions on N follow directly from the procedure itself:

$$0.75n^{1/3} \ll N < n, \qquad 0.75 \approx [2\zeta(3)]^{-1/3}. \tag{14}$$

Table 1

n	N	$p^{2\mathrm{D}}(n)$	$p_N^{2\mathrm{D}}(n)$				relative errors, %		
			exact values	calc. 1	calc. 2	calc. 3	1	2	3
10	9	500	458	474	498	497	3.5	8.8	8.7
15	14	6879	6703	6791	7082	7073	1.3	5.7	5.5
20	19	75278	74651	75003	77574	77478	0.5	3.9	3.8
	18		74161	74898	77435	77339	1.0	4.4	4.3

Number of restricted plane partitions: the exact values are from [18], the column calc. 1 corresponds to the exact $p^{2\mathrm{D}}(n)$ from [19], and the columns calc. 2 and calc. 3 are based on $p^{2\mathrm{D}}(n)$ in (14) with the constant c respectively taken from [8] and [20], [21].

Some calculation results with Eq. (13) are given in Table 1. The exact numbers of unrestricted plane partitions $p^{2\mathrm{D}}(n)$ can be found in [19], while the asymptotic dependence on n is given by [20], [21]

$$p^{2\mathrm{D}}(n) = \frac{[2\zeta(3)]^{7/36}}{\sqrt{6\pi}}\,n^{-25/36}\exp\biggl\{\frac{3}{2}[2\zeta(3)]^{1/3}n^{2/3}+c\biggr\}, \tag{15}$$

where $c = \zeta'(-1) = -0.165421\ldots$. The value

$$c = -\frac{1}{6} = -0.166666\ldots$$

was obtained in [8], which gives a better approximation for $n \le 7573$ but does not describe the behavior as $n \to \infty$ sufficiently well.

5. Discussion

As can be seen from Table 1, a good accuracy is achieved for the number of restricted plane partitions given by Eq. (13), at least for $n = 10, \ldots, 20$. Higher relative errors (4–9%) are mainly due to the error of the asymptotic formula (14) for the number of unrestricted plane partitions. An expected monotonic decrease of the relative error is observed at least for $N/n = 9/10$.

Further planned studies of this problem include solving recurrence relation (1) for $Z_N(x)$ numerically to verify the x and N behavior, especially in the limits as $x \to 1$ and with large N. After exact values of $p_N^{\mathrm{2D}}(n)$ become available, conjectured formula (13) can be verified for larger n.

It seems tempting to extend the suggested approach to higher-dimensional partitions, but we refrain from doing so before restricted plane partitions are thoroughly analyzed.

Acknowledgments. This work was supported in part by the Ministry of Education and Science of Ukraine (Grant ФФ-110Ф, No. 0112U001275).

REFERENCES

1. N. M. Bogoliubov, *Theor. Math. Phys.*, **150**, 165–174 (2007).
2. G. E. Andrews, *The Theory of Partitions* (Encycl. Math. Its Appl., Vol. 2), Addison-Wesley, Reading, Mass. (1976).
3. G. Andrews and P. Paule, *J. London Math. Soc.* (2), **76**, 647–666 (2007).
4. F. C. Auluck and D. S. Kothari, *Proc. Cambridge Philos. Soc.*, **42**, 272–277 (1946).
5. V. S. Nanda, *Proc. Cambridge Philos. Soc.*, **47**, 591–601 (1951).
6. S. Grossmann and M. Holthaus, *Phys. Rev. Lett.*, **79**, 3557–3560 (1997).
7. M. N. Tran, M. V. N. Murthy, and R. K. Bhaduri, *Ann. Phys.*, **311**, 204–219 (2004).
8. D. Prokhorov and A. Rovenchak, *Condens. Matter Phys.*, **15**, 33001 (2012); arXiv:1210.1415v1 [cond-mat.stat-mech] (2012).
9. C. S. Srivatsan, M. V. N. Murthy, and R. J. Bhaduri, *Pramana – J. Phys.*, **66**, 485–494 (2006).
10. A. Rovenchak, *Low Temp. Phys.*, **35**, 400–403 (2009).
11. R. P. Stanley, *Stud. Appl. Math.*, **50**, 167–188, 259–279 (1971).
12. N. M. Bogoliubov, *J. Phys. A*, **38**, 9415–9430 (2005).
13. P. Borrmann and G. Franke, *J. Chem. Phys.*, **98**, 2484–2485 (1993).
14. R. N. Bracewell, *The Fourier Transform and its Application*, McGraw-Hill, New York (2000).
15. C. Weiss and M. Holthaus, *Europhys. Lett.*, **59**, 486–492 (2002).
16. P. Erdös and J. Lehner, *Duke Math. J.*, **8**, 335–345 (1941).
17. P. A. MacMahon, *Phil. Trans. R. Soc. London A*, **187**, 619–673 (1897).
18. On-line Encyclopedia of Integer Sequences, "Triangle read by rows: $T[n,k]$ = plane partitions of n containing k parts," http://oeis.org/A091298 (2014).
19. On-line Encyclopedia of Integer Sequences, "Number of planar partitions of n," http://oeis.org/A000219 (2014).
20. L. Mutafchiev and E. Kamenov, *C. R. Acad. Bulgare Sci.*, **59**, 361–366 (2006).
21. E. M. Wright, *Quart. J. Math. Oxford Ser.*, **os-2**, 177–189 (1931).