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1997 Russ. Math. Surv. 52 379

(http://iopscience.iop.org/0036-0279/52/2/A13)

Russian Math. Surveys **52**:2 379–386
Uspekhi Mat. Nauk **52**:2 139–146 UDC 519.2

In memory of R. L. Dobrushin

Limit distribution of the energy of a quantum ideal gas from the viewpoint of the theory of partitions of natural numbers

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R. L. Dobrushin was one of those people who could not only listen to 'someone else's' problems but also find in them a connection with his own problems. On many occasions I discussed with him topics close to the one below. Conversations with him were always interesting and helpful. Dobrushin's mathematical thinking, which was above all probabilistic, made it possible for him to see clearly the strictly probabilistic part of the problem, which is always present in almost every asymptotic problem. We would often find something in common when discussing questions far removed from mathematics. His colourful personality and mathematical talent attracted friends and students. Dobrushin himself was certainly among those who significantly determined the general picture of our mathematics.

0. When posing and investigating problems on the asymptotic behaviour and limit form of combinatorial and geometric objects the author often looks back to statistical physics. Many problems which give rise to such questions (random partitions, substitutions, asymptotic representation theory, configuration growth, statistical geometry, number theory, and the like) have an unambiguous statistical character. These problems have diverse motivation and various answers and connections with other branches of mathematics. At the same time, apart from their intrinsic interest, they obviously should play the role of useful algebraic and geometric models for more complex problems (such as phase transitions in many-dimensional systems, and the like).

The goal of the present paper is to show how close to one another the following two problems are: the typical asymptotic form of the partition of a natural number and computing the limit distribution of the energy of a quantum ideal gas. We present the solution to the first problem and interpret it in terms of the other. There is no doubt about the usefulness of the parallelism in both domains.

1. Remarkably, the problem of energy distribution in the grand canonical ensemble and the microcanonical ensemble for a quantum ideal gas with some statistics belongs to a large class of problems including the problems of additive asymptotic number theory with the number of components growing unboundedly, and, in particular, the problem of partitions of natural numbers and vectors. In the traditional theory one is usually interested in the number or the asymptotic behaviour of the number of solutions (partitions). But we are also interested in the more general question of the asymptotic structure of typical partitions or configurations, their limit shape, invariants, and so on. As a rule, in additive number theory (for example, in the classical Waring problem) the number of components is fixed. But in problems in combinatorics, geometry, not to mention statistical physics, the number of components grows in a prescribed or random way.

Here is one formulation of this kind. We consider a countable set $\Lambda \subset \mathbb{R}_+$ without finite limit points and a number $E \in \mathbb{R}_+$. A measure (statistics) is given on the set of all partitions $E = \lambda_1 + \lambda_2 + \cdots$, $\lambda_i \in \Lambda$, of E with decreasing order $\lambda_1 \geqslant \lambda_2 \geqslant \dots$. We can pose the question of the asymptotic properties of such measures as $E \to \infty$. Do they have a limit in some sense (after the necessary normalization of partitions)? If so, is it degenerate or not? That is, are almost all partitions asymptotically the same?

We can replace $\mathbb{R}_+$ by $\mathbb{R}_+^n$ or by a commutative semigroup, and Λ by a subset of the semigroup. One can impose various additional conditions on the components, and the like (the growth of their number, dimensions, and so on). The problem of the asymptotic behaviour of measures and, in particular, of the limit shape is a key problem connected with many branches of mathematics, see [1], [2].

2. We shall consider in more detail some problems concerned with *partitions* of natural numbers. In this case $\Lambda = \mathbb{N}$ and $E \equiv n \in \mathbb{N}$ in the notation above. We shall denote by $\mathcal{P}_n = \{\lambda : \lambda \vdash n\}$ the set of partitions of a natural number n and by $n(\lambda) = n$ the sum for a given partition λ. We also put $\mathcal{P} = \bigcup_n \mathcal{P}_n$. For a given partition $\lambda = \{\lambda_i\}$ we put $r_k(\lambda) = \#\{i : \lambda_i = k\}$ (the number of components equal to a given number). We call these *occupation numbers*. It is obvious that $\sum_k kr_k(\lambda) = n(\lambda)$ and the number of components is $\sum_k r_k(\lambda) \equiv N(\lambda)$. We denote by $\mathcal{P}_{n,N} = \{\lambda : n(\lambda) = n,\ N(\lambda) = N\}$ the number of partitions of n with N components. The function $\varphi_\lambda \colon \mathbb{R}_+ \to \mathbb{N}$ given by

$$\varphi_\lambda(t) = \sum_{k=[t]}^{\infty} r_k(\lambda)$$

will be called the *distribution of a partition* λ. Its subgraph is the *Young diagram*. It is obvious that $\varphi_\lambda(0) = N(\lambda)$ and $\int_0^\infty \varphi_\lambda(t)\,dt = n(\lambda)$.

Let us distinguish an important class of measures on partitions: we say that a family of statistics (measures) μ_x on $\mathcal{P}$ is *multiplicative* if

1) the occupation numbers $\{r_k(\,\cdot\,)\}_{k=0}^{\infty}$ as functions on the measure spaces $(\mathcal{P}, \mu_x)$ are independent for all μ_x;
2) the restriction of μ_x to $\mathcal{P}_n$, $n = 1, 2, \dots$, is independent of x:

$$\frac{1}{\mu_x(\mathcal{P})}\mu_x\big|_{\mathcal{P}_n} \equiv \mu^n.$$

Lemma. *Every multiplicative family can be reduced to the following form by a change of parameter*: x *runs through an open interval* $(0, R_0)$ *with* $R_0 \leqslant \infty$; *there are functions* $\mathcal{F}_k(\,\cdot\,)$ *analytic in the disc* $|z| < R_0$ *that have non-negative Taylor coefficients*, $\mathcal{F}_k(y) = \sum_{r=0}^{\infty} c_{kr} y^r$, $k = 1, 2, \dots$, *with* $c_{kr} \geqslant 0$, *and*

$$\mu_x(\{\lambda\}) = x^{n(\lambda)} \prod_k \mathcal{F}_k(x^k)^{-1} c_{kr_k(\lambda)},$$

$$\mu_x(\{\lambda\} : r_k(\lambda) = r) = c_{kr} x^{kr} \mathcal{F}_k(x^k)^{-1}.$$

Remarks. 1. Comparison of the above formulae implies that $r_1(\lambda), r_2(\lambda), \dots$ are independent functions.

2. The lemma asserts that a multiplicative family can be defined by one function $\mathcal{F}(x)$ analytic in the disc $|z| < R_0$ and its decomposition into an infinite product $\mathcal{F}(x) = \prod_{k=1}^{\infty} \mathcal{F}_k(x^k)$, where all the factors $\mathcal{F}_k$ have non-negative Taylor coefficients. $\mathcal{F}_k$ is the generating function of the distribution of $r_k(\,\cdot\,)$.

3. The normalized restrictions of μ_x to $\mathcal{P}_n$ look like this:

$$\mu^n(\{\lambda\}) \equiv \frac{1}{\mu_x(\mathcal{P}_n)}\mu_x(\{\lambda\}) = Q_n^{-1}\prod_k c_{kr_k(\lambda)},$$

where $Q_n = \sum_{\lambda\vdash n}\prod_k c_{kr_k(\lambda)}$, μ_x being a convex combination of the measures μ^n:

$$\mu_x = \sum_{n=0}^{\infty} x^n Q_n \mathcal{F}(x)^{-1}\mu^n.$$

The same data $\big(\mathcal{F}(x) = \prod_k \mathcal{F}_k(x^k)\big)$ define a multiplicative family on the microcanonical ensemble $\mathcal{P}_{n,N}$. In this case the formulae are

$$\mu_{x,z}\big(\{\lambda\}\big) = x^{n(\lambda)}\cdot z^{N(\lambda)}\cdot\prod_k \mathcal{F}_k(zx^k)^{-1}c_{kr_k(\lambda)};$$

$$\mu^{n,N}(\{\lambda\}) \equiv \frac{1}{\mu_{x,z}(\mathcal{P}_{n,N})}\mu_{x,z}(\{\lambda\}) = Q_{n,N}^{-1}\prod_k c_{kr_k(\lambda)},$$

where

$$Q_{n,N} = \sum_{\substack{\lambda\vdash n\\ N(\lambda)=N}}\prod_k c_{kr_k(\lambda)} \qquad \text{(micropartition function)}.$$

Here the analogies with the notions of statistical physics of an ideal gas are obvious, see below.

Now the basic problem can be stated like this: *to find the asymptotic behaviour of μ^n, $\mu^{n,N}$ as $n, N \to \infty$ and of μ_x, $\mu_{x,z}$ as $x \to R_0$ and $z \to 1$ and to compare them.* The equality of the limits of these measures means that the ensembles $(\mathcal{P}, \mu_x)$ and $(\mathcal{P}_n, \mu^n)$ or $(\mathcal{P}, \mu_{x,z})$ and $(\mathcal{P}_{n,N}, \mu^{n,N})$ are asymptotically equivalent. The question of asymptotic behaviour involves finding a suitable *non-trivial normalization* (*scaling*), which makes it possible to consider all measures on the same scale. The scaling is essentially unique and a direct answer to the above question will be given below in the case of the classical statistics for a quantum ideal gas and their generalizations.

3. From the whole class of multiplicative measures we shall consider only the following:

$$\mathcal{F}^{BE}_{\{b_k\}}(x) = \prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{b_k}}, \qquad \mathcal{F}^{FD}_{\{b_k\}}(x) = \prod_{k=1}^{\infty}(1+x^k)^{b_k},$$

where $b_k \in \mathbb{N}$, $k = 1, 2, \dots$. By virtue of what we said earlier, these functions define statistics on $\mathcal{P}$, $\mathcal{P}_n$, $\mathcal{P}_{n,N}$, namely, $\mu_{x,\{b_k\}}$, and so on. We shall omit the index $\{b_k\}$

whenever possible. If $b_k = j_d(k) = \#\{(k_1, \dots, k_d) \in \mathbb{Z}^d : k_1^2 + \dots + k_d^2 = k\}$, then these statistics are the same as the Bose–Einstein and Fermi–Dirac statistics of a d-dimensional quantum ideal gas.

We define $\alpha = \alpha(\{b_k\})$ by

$$\alpha = \inf\Big\{\alpha' : \text{the Dirichlet series } \sum_{k=1}^{\infty} \frac{b_k}{k^s} \text{ is an analytic function of } s \text{ on the half-plane } \operatorname{Re} s \geqslant \alpha'\Big\}.$$

By the well-known Siegel theorem, $\alpha = d/2$ for $b_k = j_k(d)$.

The parameter α is the only characteristic of $\mathcal{F}^{BE}$ and $\mathcal{F}^{FD}$ that is needed to answer the questions posed above. We shall assume that $\alpha < \infty$.

The following theorem asserts that there is a unique scaling such that μ_x and μ^n are equal to one another and the limit is a degenerate measure concentrated on one special curve uniquely determined by α.

Theorem. *For any $a, b > 0$ and $\varepsilon > 0$*

$$\lim_{x\to 1} \mu_x \left\{ \lambda \in \mathcal{P} : \sup_{t\in[a,b]} \left| n^{-\frac{\alpha}{\alpha+1}} \sum_{k \geqslant t n^{\frac{1}{1+\alpha}}} r_k(\lambda) - \int_t^{\infty} u^{\alpha-1} \frac{e^{-cu}}{1 \mp e^{-cu}}\, du \right| < \varepsilon \right\} = 1,$$

$$\lim_{n\to \infty} \mu^n \left\{ \lambda \vdash n : \sup_{t\in[a,b]} \left| n^{-\frac{\alpha}{\alpha+1}} \sum_{k \geqslant t n^{\frac{1}{1+\alpha}}} r_k(\lambda) - \int_t^{\infty} u^{\alpha-1} \frac{e^{-cu}}{1 \mp e^{-cu}}\, du \right| < \varepsilon \right\} = 1.$$

The choice of the sign $\mp$ corresponds to the BE and FD statistics, respectively, the constant c being defined below.

Thus, the scaling is defined as follows up to a constant: $\big(n^{\frac{1}{\alpha+1}}, n^{\frac{\alpha}{\alpha+1}}\big)$. The limit measures on the two ensembles coincide and are equal to the δ-measure on the curve defined by

$$\Gamma_\alpha(t) = \int_t^{\infty} u^{\alpha-1} \frac{e^{-cu}}{1 \mp e^{-cu}}\, du. \tag{$*$}$$

$\Gamma_\alpha(t)$ is a distribution density, so c is determined by the condition

$$\int_0^{\infty} \Gamma_\alpha(t)\, dt = 1.$$

We observe that $\Gamma_\alpha(0) < \infty$ for $\alpha > 1$ (that is, $d > 2$) and $\Gamma_\alpha(0) = \infty$ for $0 < \alpha \leqslant 1$. The case $\alpha = 1$ corresponds, on the one hand, to the uniform distribution on $\mathcal{P}_n$ and the Euler functions

$$\mathcal{F}^{BE}(x) = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}, \qquad \mathcal{F}^{FD}(x) = \prod_{k=1}^{\infty} (1 + x^k),$$

and, on the other hand, to the two-dimensional ideal gas ($\alpha = d/2$). The curve $\Gamma_1(t)$ in symmetric form is given by the equation (the BE case)

$$\exp\left(-\frac{\pi}{\sqrt{6}}x\right) + \exp\left(-\frac{\pi}{\sqrt{6}}y\right) = 1,$$

the scaling being $(\sqrt{n}, \sqrt{n})$.

Interestingly, its behaviour at zero (logarithmic singularity) explains an old theorem of Erdös that in a typical partition of a number n the number of components grows as $c\sqrt{n}\ln n$, which is directly related to the question of phase transition (more precisely, its absence when $d \leqslant 2$, see below).

Fixing the growth of the number of components leads to a similar theorem for the measures $\mu^{n,N}$, namely, if $N = vn^{\frac{\alpha}{1+\alpha}}$ with $v > 0$, then the corresponding formula for $\Gamma_{\alpha,v}$ is

$$\Gamma_{\alpha,v}(t) = \int_t^\infty u^{\alpha-1}\frac{e^{-cu}}{1 \mp ve^{-cu}}\,du$$

when $\alpha \leqslant 1$. If $\alpha > 1$, then the graph of the curve is supplemented by an atom at 0 ('Bose–Einstein condensation', see below) when v is large enough. The curves Γ_α and $\Gamma_{\alpha,v}$ are the limits of Young diagrams with suitable normalization, that is, the limit distributions of partitions. The functions Γ_α and $\Gamma_{\alpha,v}$ contain information on the leading term of the asymptotics of any functional of a typical partition.

4. We shall briefly consider the basic question of the connection with the statistics of an ideal quantum gas. L. A. Khalfin's constructive criticism was useful to the author and helped to refine some of the relationships.

In this case the state (= a point of the ensemble = configuration) is determined by the momenta of the particles in the configuration, that is, by the eigenvalues (with multiplicities) of the Dirichlet problem for the Laplace operator with periodic boundary conditions. Let V be the volume of the d-dimensional torus and ω the state of the system. Then the energy of the state is given by

$$E_\omega = \frac{(2\pi\hbar)^2}{2m}\sum_{q\in\omega\subset\mathbb{Z}^d}\frac{1}{V^{\frac{2}{d}}}\|q\|^2,$$

where $\|\cdot\|$ is the d-dimensional Euclidean norm. Adopting a system of units in which the factor in front of the sum is equal to 1, we obtain to within the notation the situation considered above, where

$$\mathcal{F}(x) = \prod_{k=1}^{\infty}\frac{1}{(1-x^k)^{j_d(k)}}$$

(the BE-statistics).

The asymptotic behaviour of all quantities (the number of particles, volume) is interrelated as the energy increases, the latter being a basic parameter for us. Perhaps in this lies some methodological difference as compared to what is traditional. Roughly speaking, energy is the natural number that we decompose: $E = n$.

The components are, to within a multiplicative constant (which involves the inverse temperature), natural numbers divided by some power of the volume.

The above theorem implies that the scaling along the axes is such that, to within a constant,

$$\left(n^{\frac{1}{\alpha+1}}, n^{\frac{\alpha}{\alpha+1}}\right) = \left(E^{\frac{2}{d+2}}, E^{\frac{d}{d+2}}\right).$$

We recall that $\alpha = d/2$. Therefore $V = V(E) \simeq E^{\frac{d}{d+2}}$ and $N = N(E) \simeq E^{\frac{d}{d+2}}$ (to within an order of magnitude). Here we consider the case when the number of particles is random rather than fixed (the chemical potential being equal to zero). The scaling constant is irrelevant to the measures μ^n, but it is essential for the scaling of μ_x (see below). It depends on x, and x in the statistical interpretation is the exponent of the inverse temperature with minus sign, divided by the appropriate power of the volume.

A simple calculation demonstrates that the relationship between the volume (that is, between the magnitude of the components or, in other words, the energy of particles) on the one hand and the number of particles on the other hand becomes automatically what it ought to be in the thermodynamical limit. Thus, *our passage to the limit corresponds exactly to the thermodynamical limit*, in which N and V have the same growth order. But if the limit of the ratio of the number of particles (components) to the volume is fixed, then we arrive at a similar problem for the conditional distribution, that is, for the microcanonical ensemble, see below.

We observe that in number-theoretic and geometric problems it is appropriate to consider situations when the relationship between $V = V(E)$ and $N = N(E)$, that is, between the number and magnitude of the components or the sum, is not necessarily as rigid as in the thermodynamical limit. For example, the number of components may be a function of the sum prescribed in advance.

Let us go back to the interpretation of the above theorem in statistical physics. Suppose that $r_k(\omega)$ is the number of particles with energy equal to k, that is, the sum of ordinary occupation numbers over a sphere of radius k in the momentum lattice. We shall state the theorem in the appropriate terms. Let $\varepsilon > 0$.

Theorem. *For any sufficiently large energy $E > E_\varepsilon$ the set of states of an ideal BE-gas (with random number of particles) for which the following property $(*)$ holds has Gibbsian measure larger than $1 - \varepsilon$:*

$$\sup_{0<a\leqslant t\leqslant b<\infty} \left| E(\omega)^{-\frac{d}{d+2}} \cdot \sum_{k \geqslant tE^{\frac{2}{d+2}}} r_k(\omega) - \Gamma_{\frac{d}{2}}(t) \right| < \varepsilon. \tag{$*$}$$

It follows that $\Gamma_{\frac{d}{2}}$ is the graph of the limit distribution density of $E(\omega)$, the energy of a typical configuration versus the energy of particles normalized to $E^{\frac{2}{d+2}}$.

It follows, for example, that in a typical configuration the total energy of all particles, each having energy in the interval $(aE^{\frac{2}{d+2}}, bE^{\frac{2}{d+2}})$, is asymptotically equal to $\int_a^b \Gamma_\alpha(t)\, dt \cdot E$ and the number of particles with energy greater than $tE^{\frac{2}{d+2}}$ is equal to $\Gamma_\alpha(t) E^{\frac{2}{d+2}}$.

The question of whether $\Gamma_\alpha(0)$ is finite or not is crucial: if it is finite, then the number of particles (components) in a typical configuration increases as $\Gamma_\alpha(0) E^{\frac{2}{d+2}}$, but if it is infinite, then slightly faster than that (by a power of the logarithm). We observe that all the formulae are also valid for fractional dimensions d. In this case $b_k = [k^{\alpha-1}]$ if $1 \leqslant \alpha$ and, as before, $d = 2\alpha$. The value of our density at zero is finite for $d > 2$.

Let us state a more explicit formula for $d = 3$, the three-dimensional ideal BE-gas:

$$\Gamma_{\frac{3}{2}}(t) = \int_t^{\infty} \sqrt{u}\,\frac{e^{-cu}}{1-e^{-cu}}\,du, \qquad c = \left[\Gamma\Big(\frac{5}{2}\Big)\zeta\Big(\frac{5}{2}\Big)\right]^{\frac{2}{5}}.$$

Here Γ is the gamma-function and ζ is the Riemann zeta-function. The distribution density $\Gamma_\alpha(t)$ is bounded at zero.

We have already touched upon the dimension $d = 2$ ($\alpha = 1$). This case is exactly the same as the classical uniform distribution on partitions. The function $\Gamma_1(\,\cdot\,)$ has been described above. Its value at zero is unbounded. The asymptotic behaviour of many concrete functionals has been found by Erdös and his successors, but they have not been mentioned in connection with statistical physics, nor has the problem of the limit shape been posed. The Erdös theorem on the typical number of components in a uniformly distributed partition of a natural number ($c\sqrt{n}\,\ln n$) is a theorem on the typical number of particles, that is, on the asymptotic behaviour of Γ_α at zero. It can also be derived from the theorem on the limit shape.

From the point of view of number theory the one-dimensional case $d = 1$ is also interesting, but obviously not so in statistical physics. Here $\alpha = \frac{1}{2}$ and $b_k = 1$ if k is a perfect square and $b_k = 0$ otherwise. This case corresponds to the problem of the limit density of the partition of a natural number into the sum of squares with an unbounded number of terms. In other words, this is Waring's unbounded square problem. Here the answer is also given by the formula stated in the theorem for $\alpha = \frac{1}{2}$.

Dimension $d = 4$ with $\alpha = 2$ is very interesting in combinatorics. This is the asymptotic theory of plane partitions or three-dimensional Young diagrams. This is dealt with in another paper.

If we fix the growth of the number of components (particles) in a natural range *to within a constant*, namely, $N \sim cE^{\frac{d}{d+2}}$ (and then, as we have already observed, $\frac{N}{V} \to v$ is also fixed automatically, v being a finite density), then the answer can also be obtained by similar methods. Here we are concerned with the microcanonical ensemble and weak limits under a suitable scaling of $\mu_{x,z}$ and $\mu^{n,N}$. In this case the chemical potential is different from zero. The limits of these measures may be different even if the relationships between the joint growth of N, n and the joint rate of convergence of x, z to 1 are compatible, see [3]. The weak limit of $\mu^{n,N}$ is again a δ-measure concentrated either on a curve or on a curve with an additional atom at zero (see above).

The difference between dimensions (including fractional ones) $d > 2$ and $d \leqslant 2$ (that is, between $\alpha > 1$ and $\alpha \leqslant 1$) is manifested in the presence of *Bose–Einstein condensation*, the essence of which can be explained in our terms as follows. If $\Gamma_\alpha(0) = \infty$ (which is the case for $d \leqslant 2$), then any given finite number of particles, that is, components (normalized relative to the appropriate power of energy) will have a lower order of magnitude than in the case when it is a random number.

Indeed, as has been observed above, for a typical configuration the latter increases slightly faster than $E^{\frac{d}{d+2}}$ as the energy E increases, which means in fact that $\Gamma_\alpha(0)$ is infinite. In this case the limit of $\mu^{n,N}$ under the appropriate scaling is concentrated on a curve, namely, the energy distribution density, the formula for which has been given. But if $\Gamma_\alpha(0) < \infty$ (that is, $d > 2$), then for any given normalized number of particles (components) larger than $\Gamma_\alpha(0)$ the 'forbidden' number will be larger than in the case when the number of particles is random, the 'shortage' of the number of particles can only be covered by particles with zero momentum, that is, by zero components, and the limit of $\mu^{n,N}$ will be concentrated on a curve with an atom at zero. This is what Bose–Einstein condensation is all about: the contribution of zero particles with zero momentum (zero components) is positive.

This explanation differs slightly from the usual ones (see [4]) by the role played by the case of zero chemical potential (random number of particles): we compare the values of $\Gamma_\alpha(0)$ and the normalized number of particles under fixed density. We recall that the temperature appears implicitly in the normalization of the number of particles, and so of the function $\Gamma_\alpha(\cdot)$ and its value at zero (see above, where scaling has been defined to within a constant). Therefore the above discussion eventually leads to the usual criterion: the condensation parameter is determined by the same known expression depending on the density and temperature (see, for example, [4], formula (55.5)), which should be compared with our $\Gamma_\alpha(0)$.

On the other hand, in the above considerations and, in particular, in the explanation of Bose–Einstein condensation, we have used nothing but combinatorial terms connected with the partition of natural numbers. Therefore, Bose–Einstein condensation can be 'observed' both in statistical physics and in the present and similar combinatorial-numerical problems. I believe that there are many such mutually beneficial intersections.

The curves Γ_α are extremals of certain variational problems. The corresponding variational principle was formulated by the author (see [1], [2]), but it will be described in more detail elsewhere. The variational principle is closely related to the method of large deviations and entropy. Dobrushin was one of the pioneers in applying these methods in statistical physics.

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Received 15/SEP/96
Translated by T. J. ZASTAWNIAK

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX