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The Infinite-State Potts Model and Restricted Multidimensional Partitions of an Integer

F. Y. Wu
Department of Physics, Northeastern University
Boston, MA 02115, U.S.A.
fywu@neu.edu

Abstract—It is shown that the partition function of the q-state Potts model on a finite d-dimensional hypercubic lattice in the $q \to \infty$ limit is precisely the generating function of $(d-1)$-dimensional restricted partitions of an integer. For $d = 2, 3$, this equivalence leads to closed-form expressions of the $q = \infty$ Potts partition function. Our discussion also establishes symmetry and reciprocal properties for the generating function of restricted partitions in higher dimensions.

Keywords—The Potts model, Restricted partitions.

1. INTRODUCTION

An outstanding unsolved problem in statistical mechanics is the q-state Potts model in d dimensions [1,2]. For a review on the Potts model and its physical relevance, see [2]. Except in the case of $q = 2$ for which the Potts model is the Ising model and soluble in $d = 2$ (see [3]), the Potts model is generally intractable. Here we establish a connection of the $q = \infty$ Potts model in d dimensions with $(d-1)$-dimensional restricted partitions of an integer. The connection enables us to solve the $q = \infty$ Potts model for $d = 2, 3$; it also leads to certain symmetry relations for the generating function of restricted multidimensional partitions.

Consider a d-dimensional hypercubic lattice of size $L_1 \times L_2 \cdots \times L_d$ with lattice sites specified by coordinates $\{n_1, n_2, \ldots, n_d\}$, $n_i = 1, 2, \ldots, L_i$, $i = 1, 2, \ldots, d$. Introduce an extra site connected by edges to *every* site in the d $(d-1)$-dimensional hyperplanes $n_i = L_i$. Particularly, the extra site is connected to the site $\{L_1, L_2, \ldots, L_d\}$ by d edges. We regard the resulting lattice as a graph $\mathcal{L}$, and there are altogether $N + 1$ vertices and Nd edges in $\mathcal{L}$, where

$$N = L_1 L_2 \cdots L_d. \tag{1}$$

An example of $\mathcal{L}$ for $d = 2$, $L_1 = 4$, $L_2 = 3$, $N = 12$ is shown in Figure 1.

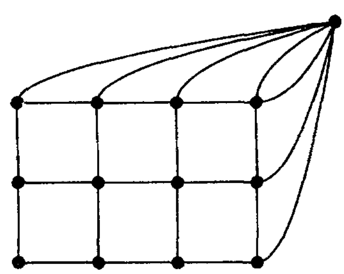

Figure 1. A graph $\mathcal{L}$ with 13 vertices and 24 edges.

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Consider the q-state Potts model on $\mathcal{L}$, with each vertex of $\mathcal{L}$ occupied by a spin which can be in q different states. Let two neighboring spins in states $\sigma, \sigma' = 1, 2 \ldots, q$ interact with an energy

$$\begin{aligned} E(\sigma, \sigma^*) &= -J, \qquad \sigma = \sigma', \\ &= 0, \qquad \sigma \neq \sigma'. \end{aligned} \tag{2}$$

In statistical mechanics one is interested in evaluating the partition function

$$Z = \sum_{\sigma_i=1}^{q} \prod_{\text{edge}} e^{-E(\sigma,\sigma')/kT}, \tag{3}$$

where k is the Boltzmann constant, T the temperature, the summations are taken over the spin states at all vertices i, and the product taken over all edges of $\mathcal{L}$.

Using the identity

$$e^{K\delta(\sigma,\sigma')} = 1 + \left(e^K - 1\right)\delta(\sigma, \sigma'), \tag{4}$$

where $K = J/kT$, one can rewrite the partition function (3) as a graphical expansion [2]. This leads to the expression

$$Z = \sum_{G \subseteq \mathcal{L}} \left(e^K - 1\right)^{e(G)} q^{n(G)}, \tag{5}$$

where the summation is taken over all subgraphs $G \subseteq \mathcal{L}$, $e(G)$ is the number of edges in G, and $n(G)$ the number of connected clusters in G, including isolated vertices. Here, a subgraph of $\mathcal{L}$ contains the same set of vertices as $\mathcal{L}$ and a subset of edges of $\mathcal{L}$.

Introducing

$$x = \left(e^K - 1\right) q^{-1/d}, \tag{6}$$

we can rewrite the partition function (5) as

$$Z_N(q, x) = \sum_{G \subseteq \mathcal{L}} q^{n(G)+e(G)/d} x^{e(G)}. \tag{7}$$

We consider $Z_N(q, x)$ in the limit of $q \to \infty$ with x fixed.

It is clear that for the subgraph $G = \mathcal{L}$ we have $n(\mathcal{L}) = 1$, $e(\mathcal{L}) = Nd$, and thus $n(\mathcal{L}) + e(\mathcal{L})/d = N + 1$. Any other subgraph $G \subset \mathcal{L}$ can be obtained by deleting edges in $\mathcal{L}$. Now, by deleting $e(G)$ edges (from $\mathcal{L}$) one can increase the number of clusters by at most $e(G)/d$; it follows that for any subgraph G we have

$$n(G) + \frac{e(G)}{d} \leq N + 1. \tag{8}$$

As a consequence of (8), for q large the leading term in $Z_N(q, x)$ is in the form q^{N+1} times a polynomial of degree N in x^d. This permits us to introduce a reduced partition function

$$Y_d\left(x^d\right) \equiv \lim_{q \to \infty} q^{-(N+1)} Z_N(q, x). \tag{9}$$

The subgraphs for which the equality holds in (8) and thus contributing in (9) will be described in Section 3.

2. MULTIDIMENSIONAL RESTRICTED PARTITIONS

The generating function of $(d-1)$-dimensional restricted partitions associated with $\mathcal{L}'$, a hypercubic lattice of size $L_1 \times L_2 \times \cdots \times L_d$, is (see [4])

$$G(L_1, L_2, \ldots, L_d; t) = 1 + \sum_{n=1}^{N} A_n(L_1, L_2, \ldots, L_d) t^n, \tag{10}$$

where $A_n(L_1, L_2, \ldots, L_d)$ is the number of distinct partitions of a positive integer n into sums of nonnegative integers $m(n_1, n_2, \ldots, n_{d-1})$ associated with vertices $\{n_1, n_2, \ldots, n_{d-1}\}$ of $\mathcal{L}'$, or, explicitly,

$$n = \sum_{n_1=1}^{L_1} \sum_{n_2=1}^{L_2} \cdots \sum_{n_{d-1}=1}^{L_{d-1}} m(n_1, \ldots, n_{d-1}), \tag{11}$$

such that

$$0 \le m(n_1', n_2', \ldots, n_{d-1}') \le m(n_1, n_2, \ldots, n_{d-1}) \le L_d, \tag{12}$$

whenever $n_1 \le n_1', n_2 \le n_2', \ldots, n_{d-1} \le n_{d-1}'$.

Our main result is the following proposition.

PROPOSITION. *The reduced partition function (9) is given by*

$$Y_d\left(x^d\right) = G\left(L_1, L_2, \ldots, L_d; x^d\right), \tag{13}$$

where $G(L_1, L_2, \ldots, L_d; t)$ is the generating function of restricted partitions of an integer into sums of integral parts associated with a $(d-1)$-dimensional hypercubic lattice $\mathcal{L}'$ of size $L_1 \times L_2 \times \cdots \times L_{d-1}$, with each part equal to or less than L_d.

The following properties of the generating function of multidimensional partitions, which are not immediately obvious from the definition (10), now follow from the proposition and its proof.

COROLLARY 1. *The generating function $G(L_1, L_2, \ldots, L_d; t)$ is symmetric in L_i.*

COROLLARY 2. *The generating function $G(L_1, L_2, \ldots, L_d; t)$ satisfies the reciprocal relation*

$$G(L_1, L_2, \ldots, L_d; t) = t^{L_1 L_2 \cdots L_d} G\left(L_1, L_2, \ldots, L_d; t^{-1}\right). \tag{14}$$

REMARK. Properties stated in Corollaries 1 and 2 for $d = 2$, the linear partition, can be, respectively, observed by taking the *conjugate* and the *complement* of a *Ferrers diagram* contained in an $L_1 \times L_2$ box.

3. PROOF OF THE PROPOSITION

To establish the proposition, we consider the generation of the reduced partition function by identifying terms in (7) having the factor q^{N+1}. These terms come from the subgraph having no edges with $\{n, e\} = \{N+1, 0\}$, the subgraph $G = \mathcal{L}$ with $\{n, e\} = \{1, Nd\}$, and other subgraphs obtained by perturbing from either of these two.

Starting from the subgraph $G = \mathcal{L}$ which has the weight x^{Nd}, one can generate the reduced partition function term by term by removing edges and modifying n and e systematically. Since by definition $n \ge 1$ and we have $n = 1$ to begin with, the number n can only increase and the minimum one can do is to increase n by 1, to $n = 2$. To hold the number $n + e/d$ constant, it is then necessary to decrease e by d. Therefore, one looks for vertices connecting to exactly d neighboring vertices.

For $d > 1$, there is only one such vertex in $\mathcal{L}$, i.e., the vertex $\{1, 1, \ldots, 1\}$ connected to the d vertices $\{2, 1, \ldots, 1\}$, $\{1, 2, \ldots, 1\}$, and $\{1, 1, \ldots, 2\}$. All other vertices are connected to more than d neighbors. Removing the d edges connecting to $\{1, 1, \ldots, 1\}$, one creates a subgraph $\mathcal{L}_1$ consisting of one isolated vertex with $\{n, e\} = \{2, (N-1)d\}$. This generates a correction factor x^{-d} to the fully covered weight x^{Nd}, and hence a term $x^{(N-1)d}$ in the reduced partition function (9).

In a similar fashion, to generate subgraphs with $n = 3$ while holding $n + e/d$ constant, one looks in $\mathcal{L}_1$, in which the vertex $\{1, 1, \ldots, 1\}$ is isolated, for vertices which are connected to exactly d neighboring vertices. There are now exactly d such vertices, namely, the d vertices originally connected to $\{1, 1, \ldots, 1\}$. All other vertices are connected to more than d neighbors.

By removing the d edges connected to any of these vertices, one finds the next term in the reduced partition function having $\{n, e\} = \{3, (N-2)d\}$. The resulting subgraphs $\mathcal{L}_2$ now have two isolated vertices contributing to a term $dx^{(N-2)d}$ in (9).

Continuing in this fashion, the process of generating terms in (9) is seen to be the same as generating subgraphs containing isolated vertices by removing d edges at a time. Denote subgraphs containing n isolated vertices constructed in this manner by $\mathcal{L}_n$, and let there be $c_n(L_1, L_2, \ldots, L_d)$ such subgraphs. For example, we have $c_1 = 1$, $c_2 = d$, and $c_3 = d(d+1)/2$ if $L_i \geq 3$. Thus, we have

$$\begin{aligned} Y_d\left(x^d\right) &= x^{Nd}\left[1 + \sum_{n=1}^{N} c_n(L_1, L_2 \ldots, L_d)x^{-nd}\right] \\ &= 1 + \sum_{n=1}^{N} c_n(L_1, L_2, \ldots, L_d)x^{nd} \\ &= x^{Nd} Y_d\left(x^{-d}\right). \end{aligned} \tag{15}$$

Here, the second equality in (15) is established by considering generating $Y_d(x^d)$ by perturbing from the subgraph with no edges, and the last line in (15) is obtained by combining the first two lines.

The proposition now follows from the second line of (15) and the identity

$$c_n(L_1, L_2, \ldots, L_d) = A_n(L_1, L_2, \ldots, L_d). \tag{16}$$

To establish (16), one observes that, by considering the set of the d integers

$$\{n_1, n_2, \ldots, n_{d-1}, m(n_1, n_2, \ldots, n_{d-1})\}, \tag{17}$$

where

$$m(n_1, n_2, \ldots, n_{d-1}) = \max\{m \mid (n_1, n_2, \ldots, n_{d-1}, m) \text{ is isolated}\}, \tag{18}$$

one has a bijection between subgraphs $\mathcal{L}_n$ (in which there are n isolated vertices) and the set of nonnegative integers $m(n_1, n_2, \ldots, n_{d-1})$ satisfying (11). The example of an $\mathcal{L}_n$ in the case of $d = 2$ is shown in Figure 2, and it is noted that the subgraphs $\mathcal{L}_n$ are precisely those contributing in (9) when the equality holds in (8). Furthermore, the rule of removing the d edges connected to a vertex $\{n_1, n_2, \ldots, n_d\}$ is that the d vertices $\{n_1, n_2, \ldots, n_i - 1, \ldots, n_d\}$, $i = 1, 2, \ldots, d$, are already isolated. But this is precisely a statement of the condition (12). Thus we have established the identity (16) and hence the proposition. Finally, since all L_i enter on the same footing in the Potts partition function, it follows that the generating function (10) is symmetric in L_i, and this establishes Corollary 1. In addition, the third line of (15) establishes Corollary 2.

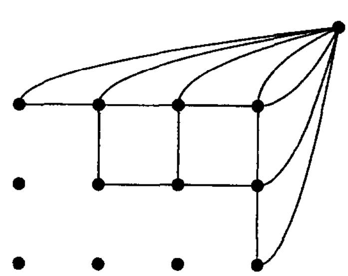

Figure 2. An example of a subgraph $\mathcal{L}_n$ for $n = 4$, with $n(\mathcal{L}_4) = 5$, $e(\mathcal{L}_4) = 16$, corresponding to the (linear) partition $\{2,1,1\}$.

4. RESULTS IN $d = 2, 3$

Now the partition generating function $G(L_i; t)$ is explicitly known for $d = 2$ as the Gaussian polynomial (see [4,5]) and $d = 3$ (see [4,6]), from which one obtains

$$Y_2(t) = \frac{(t)_{L_1+L_2}}{(t)_{L_1}(t)_{L_2}}, \tag{19}$$

$$Y_3(t) = \frac{[t]_{L_1+L_2+L_3-1}[t]_{L_1-1}[t]_{L_2-1}[t]_{L_3-1}}{[t]_{L_1+L_2-1}[t]_{L_2+L_3-1}[t]_{L_1+L_3-1}}, \tag{20}$$

where

$$(t)_L \equiv \prod_{p=1}^{L}(1-t^p), \qquad [t]_L \equiv \prod_{p=1}^{L}(t)_p. \tag{21}$$

We note that both (19) and (20) are symmetric in L_i. However, the problem of restricted partitions and hence the evaluation of $Y_d(x^d)$ remain open for $d > 3$.

Finally, we explicitly evaluate the "per-site" generating function in a thermodynamic limit defined by (see [7])

$$f_d(t) \equiv \lim_{L_1\to\infty}\lim_{L_2\to\infty}\cdots\lim_{L_d\to\infty}\left[\frac{1}{N+1}\ln Y_d(t)\right]. \tag{22}$$

This is done by noting that the zeroes of the polynomials $Y_2(t)$ and $Y_3(t)$ in the complex t plane are simply superpositions of zeroes of $(t)_L$ which are all on the unit circle $|t| = 1$. Let the zeroes of $Y_d(t)$ be located at $e^{i\theta_j}$. Then we can write

$$Y_d(t) = \prod_j \left(e^{i\theta_j} - t\right), \qquad d = 2, 3. \tag{23}$$

Now the zeroes of $(t)_L$ are distributed uniformly on the unit circle in the limit of $L \to \infty$ [7]. It follows that zeroes of $Y_2(t)$ and $Y_3(t)$ are also distributed uniformly on the unit circle. This leads to, for $d = 2, 3$,

$$f_d(t) = \frac{1}{2\pi}\int_0^{2\pi}\ln(e^{i\theta} - t)\,d\theta. \tag{24}$$

Thus, for the $d = 2, 3$ Potts model, we have from (9) the per-site free energy

$$\begin{aligned} f_d(x^d) &\equiv \lim_{N\to\infty}\lim_{q\to\infty}\left[\frac{1}{N+1}\ln Z_N(q,x) - \ln q\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\ln(e^{i\theta} - x^d) \\ &= \begin{cases} \ln|x^d|, & |x| > 1, \\ 0, & |x| < 1. \end{cases} \end{aligned} \tag{25}$$

This leads to the occurrence of a first-order transition at $x = 1$, a transition previously known to exist for $q > 4$ in $d = 2$ (see [8]) and for $q \to \infty$ in any d (see [9]).

It should be pointed out that in statistical mechanics one needs to take the limit $N \to \infty$ first before taking other limits or derivatives. In one dimension, our Potts model lattice is a chain of $N + 1$ sites for which one has

$$Z_N(q,x) = q^{N+1}(x+1)^N, \tag{26}$$

and from which one can see explicitly that the two limits in the first line of (25) commute. Assuming that the two limits also commute for $d = 2, 3$, then the expression (25) solves the $q = \infty$ Potts model.

Finally, we remark that the previously known first-order transition in the $q \to \infty$ limit [9] appears to suggest that zeroes of $Y_d(t)$ are on the unit circle in the thermodynamic limit for any d. This is a highly intriguing result [10] which is not obvious combinatorially.

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