Internat. J. Math. & Math. Sci.
Vol. 10 No. 4 (1987) 625-640

SRINIVASA RAMANUJAN (1887-1920) AND THE THEORY OF PARTITIONS OF NUMBERS AND STATISTICAL MECHANICS A CENTENNIAL TRIBUTE

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(Received June 5, 1987)

ABSTRACT. This centennial tribute commemorates Ramanujan the Mathematician and Ramanujan the Man. A brief account of his life, career, and remarkable mathematical contributions is given to describe the gifted talent of Srinivasa Ramanujan. As an example of his creativity in mathematics, some of his work on the theory of partition of numbers has been presented with its application to statistical mechanics.

KEYWORDS AND PHRASES. Partition of Numbers, Congruence Identities, Asymptotic formula, Restricted and unrestricted Partitions, Bose-Einstein statistics, Fermi-Dirac statisitics, Maxwell-Boltzmann statisitics, Gentile statistics.
1980 AMS SUBJECT CLASSIFICATION CODE. 01A70, 11P57, 11P72, 11P80, 82A05, 82A15.

1. INTRODUCTION

Srinivasa Ramanujan is universally considered as one of the mathematical geniuses of all time. He was born in India a hundred years ago on December 22 of that year. His remarkable contributions to pure mathematics placed him in the rank of Gauss, Galois, Abel, Euler, Fermat, Jacobi, Riemann and other similar stature. His contributions to the theory of numbers are generally considered unique. During his life-time, Ramanujan became a living legend and a versatile creative intellect. His name will be encountered in the history of mathematics as long as humanity will study mathematics.

Ramanujan was born on December 22, 1887 in Brahmin Hindu family at Erode near Kumbakonam, a small town in South India. His father was a clerk in a cloth-merchant's office in Kumbakonam, and used to maintain his family with a small income. His mother was a devoted housewife and had a strong religious belief. However, there was no family history of mathematical or scientific genius.

At the age of seven, young Ramanujan was sent to the high school of Kumbakonam and remained there until he was sixteen. He was soon found to be a brilliant student and his outstanding ability had begun to reveal itself before he was ten. By the time Ramanujan was twelve or thirteen, he was truely recognized as one of the most outstanding young students. He remained brilliant throughout his life and his talent and interest were singularly directed toward mathematics. Like Albert Einstein, Ramanujan became entranced by an elementary text book entitled A Synopsis of Elementary Results in Pure and Applied Mathematics by George Shoobridge Carr. No doubt that this book has had a profound influence on him and his familiarity with it marked the true starting-point of his mathematical discovery. In 1903, Ramanujan passed the Matriculation Examination of the University of Madras and joined the Goverenment College at Kumbakonam in 1904 with the Subrahmanyan Scholarship which is usually awarded to students for proficiency in Mathematics and English. At the College he used to spend most of his time studying mathematics. His consequent neglect of his other subjects resulted in his failure to get promotion to the senior class. Consequently, he lost his scholarship. He was so disappointed that he dropped out from the college. In 1906, he entered Pachaiyappa's College in Madras and appeared as a private student for the F.A. Examination in December 1907 and unfortunately again failed. He was very disappointed but continued his independent study and research in pure mathematics.

During the summer of 1909, Ramanujan married Janaki and it became necessary for him to find some permanent job. Being unemployed for about six years, he accepted a small job in 1912 at the Madras Port Trust as clerk. He has now a steady job, and he found he had enough time to do his own research in mathematics. He had already published his first paper in the Journal of the Indian Mathematical Society in the December issue of volume 3, 1911. During the next year, Ramanujan published two more papers in volume 4 (1912) of the same journal.

At the advice of his teacher and friend, Seshu Aiyar, Ramanujan wrote a letter on January 13, 1913 to famous British mathematician G.H. Hardy, then Fellow of Trinity College, Cambridge. Enclosed also in this letter was a set of mathematical results incuding one hundred and twenty theorems. After receiving this material, Hardy discussed it with J.E. Littlewood with regard to Ramanujan's mathematical talent. At the beginning Hardy was reluctant, but impressed by Ramanujan's results on continued fractions. Finally, Hardy decided to bring Ramanujan to Cambridge in order to pursue some joint research on mathematics. Ramanujan was pleased to receive an invitation from Hardy to work with him at Cambridge. But the lack of his mother's permission combined with his strong Hindu religion prejudices forced him to decline Hardy's offer. As a result of his further correspondence with Hardy, Ramanujan's talent was brought to the attention of the University of Madras. The University made a prompt decision to grant a special scholarship to Ramanujan for a period of two years. On May 1, 1913, the 25 year old Ramanujan formerly resigned from the Madras Port Trust Office and joined the University of Madras as a research scholar with a small scholarship. He remained in that position until his departure for Cambridge on March 17, 1914.

During the years 1903-1914, Ramanujan devoted himself almost entirely to mathematical research and recorded his results in his own notebooks. Before his arrival in Cambridge, Ramanujan had five research papers to his name, all of which appeared in the Journal of the Indian Mathematical Society. He discovered and/or rediscovered a large number of most elegant and beautiful formulas. Thes results were concerned with Bernoulli's and Euler's numbers, hypergeometric series, functional equation for the Riemann zeta function, definite integrals, continued fractions and distribution of primes. During his stay in Cambridge from 1914 to 1919, Ramanujan worked continually together with Hardy and Littlewood on many problems and results conjectured by himself. His close association with two great mathematicians enabled him not only to learn mathematics with rigorous proofs but also to create new mathematics. Ramanujan was never disappointed or intimidated even when some of his results, proofs or conjectures were erroneous or even false. Absolutely no doubt, he simply enjoyed mathematics and deeply loved mathematical formulas and theorems. It was in Cambridge where his genius burst into full flower and he attained great eminence as a gifted mathematician of the world. Of his thirty-two papers, seven were written in collaboration with Hardy. Most of these papers on various subjects took shape during the super-productive period of 1914-1919. These subjects include the theory of partitions of numbers, the Rogers-Ramanujan identities, hyper-geometric functions, continued fractions, theory of representation of numbers as sums of squares, Ramanujan's τ-function, elliptic functions and q-series.

In May 1917, Hardy wrote a letter to the University of Madras informing that Ramanujan was infected with an incurable disease, possibly tuberculosis. In order to get a better medical treatment, it was necessary for him to stay in England for some time more. In spite of his illness, Ramanujan continued his mathematical research even when he was in bed. It was not until fall of 1918 that Ramanujan showed any definite sign of improvement. On February 28, 1918, he was elected a Fellow of the Royal Society at the early age of thirty. He was the first Indian on whom the highest honor was conferred at the first proposal. Niel Bohr was the only other eminent scientist so elected as the Fellow of the Royal Society. On October 13, 1918, he was also elected a Fellow of the Trinity College, Cambridge University with a fellowhsip of £250 a year for the next six years. In his announcement of his election with the award, Hardy forwarded a letter to the Registrar of Madras University by saying, "He will return to India with a scientific standing and reputation such as no Indian has enjoyed before, and I am confident that India will regard him as the treasure he is." He also asked the University to make a permanent arrangement for him in a way which could leave him free for research. The University of Madras promptly responded to Hardy's request by granting an award of £250 a year for five years from April 1, 1919 without any duties or assignments. In addition, the University also agreed to pay all of his travel expenses from England to India. In the meantime, Ramanujan's health showed some signs of improvement. So it was decided to send him back home as it deemed safe for him to travel. Accordingly, he left England on February 27, 1919 and then arrived at Bombay on March 17, 1919. His return home was a very pleasant news for his family, but everybody was very concerned to see his mental and physical conditions as his body had become thin and emaciated. Everyone hoped that his return to his homeland, to his wife and parents and to his friends may have some positive impact on his recovery from illness. Despite his loss of weight and energy, Ramanujan continued his mathematical research even when he was in bed.

In spite of his health gradually deteriorating, Ramanujan spent about nine months in different places including his home town of Kumbakonam, Madras and a village of Kodumudi on the bank of the river Kaveri. The best medical care and treatment availalbe at that time were arranged for him. Unfortunately, everything was unsuccessful. He died on April 29, 1920 at the age of 32 at Chetput, a suburb of Madras, surrounded by his wife, parents, brothers, friends and admirers.

In his last letter to Hardy on January 12, 1920, three months before his death, Ramanujan wrote: "I discovered very interesting functions recently I call 'Mock' ϑ-functions. Unlike 'False'ϑ-functions (studied by Professor Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary ϑ-functions. I am sending you with this letter some examples." Like his first letter of January 1913, Ramanujan's last letter was also loaded with many interesting ideas and results concerning q-series, elliptic and modular functions. In order to pay tribute to Srinivasa Ramanujan, G.N. Watson selected the contents of Ramanujan's last letter to Hardy along with Ramanujan's five pages of notes on the Mock Theta functions for his 1935 presidential address to the London Mathematical Society. In his presidential address entitled "The Final Problem: An Account of the Mock Theta Functions", Watson (1936) discussed Ramanujan's results and his own subseqent work with some detail. His concluding remarks included: "Ramanujan's discovery of the Mock Theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight ...." Clearly, Ramanujan's contributions to elliptic and modular functions had also served as the basis of the subsequent developments of these areas in the twentieth century.

2. RAMANUJAN-HARDY'S THEORY OF PARTITIONS

As an example of Ramanujan's creativity and outstanding contribution to mathematics, we briefly describe some of his work on the theory of partitions of numbers and its subsequent applications to statistical mechanics. Indeed, the theory of partitions is one of the monumental examples of success of the Hardy-Ramanujan partnership. Ramanujan shared his interest with Hardy in the unrestricted partition function or simply the partition function $p(n)$. This is a function of a positive integer n which is a representation of n as a sum of strictly positive integers. Thus $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$ and $p(6) = 11$. We define $p(0) = 1$. Thus the map $n \to p(n)$ defines the partition function. More explicitly, the unrestricted partitions of a number 6 are given as $6=1+1+1+1+1+1= 2+2+2= 2+2+1+1 = 2+1+1+1+1=3+3= 3+2+1= 3+1+1+1= 4+2= 4+1+1= 5+1$. Hence $p(6) = 11$. There are three partitions of 6 into distinct integers: $6 = 5+1 = 4+2$. There are four partitions of 6 into odd parts: $5+1 = 3+3 = 3+1+1+1 = 1+1+1+1+1+1$. The number 6 has only one partition into distinct odd parts: $5+1$. We also note that there are 4 partitions of 6 into utmost 2 integers, and there are four partitions of six into integers which do not exceed 2. And there are 3 partitions of six into 2 integers and there are equally 3 partitions of 6 into integers with 2 as the largest.

It follows from the above examples that the value of the partition function $p(n)$ depends on both the size and nature of parts of n. These examples also lead to the concept of restricted and unrestricted partitions of an integer. The restrictions may sometimes be so stringent that some numbers have no partitions at all. For example, 10 cannot be partitioned into three distinct odd parts.

There is a simple geometric representation of partitions which is usually shown by using a display of lattice points (dots) called a Ferrer graph. For example, the partition of 20 given by 7+4+4+3+1+1 can be represented by 20 dots arranged in five rows as follows:

```
•   •   •   •   •   •   •
•   •   •   •
•   •   •   •
•   •   •
•
•
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Reading this graph vertically, we get another partition of 20 which is 6+4+4+3+1+1+1. Two such partitions are called conjugate. Observe that the largest part in either of these partitions is equal to the number of parts in the other. This leads to a simple but interesting theorem which states that the number of partitions of n into m parts is equal to the number of partitions of n into parts with m as the largest part. Several theorems can be proved by simple combinatorial arguments involving graphs.

Above examples with the geometrical representation indicate that partitions have inherent symmetry. In quantum mechanics, such geometric representations of partitions are known as Young Tableaux which was introduced by Young for his study of symmetric groups. They were also found to have an important role in the analysis of the symmetries of many-electron systems.

The above discussions also illustrate some important and useful role of the partition function from mathematical, geometrical and physical points of view. In additive questions of the above kind it is appropriate to consider a power series generating function of p(n) defined by

$$F(x) = \sum_{n=0}^{\infty} p(n)\, x^n, \qquad |x| < 1 \tag{2.1}$$

From this elementary idea of generating function, Euler formulated the analytical theory of partitions by proving a simple but a remarkable result:

$$F(x) = \prod_{m=1}^{\infty} (1-x^m)^{-1} = \sum_{n=o}^{\infty} p(n)\, x^n, \; |x| < 1 \tag{2.2}$$

where $p(o) = 1$

If $0 \leq x < 1$ and an integer $m > 1$ and

$$F_m(x) = \prod_{k=1}^{m} (1 - x^k)^{-1} = 1 + \sum_{n=0}^{\infty} p(n)\, x^n, \; |x| < 1, \tag{2.3}$$

then it can be proved that

$$p_m(n) \leq p(n), \; p_m(n) = p(n), \; 0 \leq n \leq m, \tag{2.4ab}$$

and

$$\lim_{m \to \infty} p_m(n) = p(n) \tag{2.5}$$

Furthermore

$$\lim_{m \to \infty} F_m(x) = F(x) \tag{2.6}$$

Euler's result (2.2) gives a generating function for the unrestricted partition of an integer n without any restriction on the number of parts or their properties such as size, parity, etc. Hence the generating function for the partition of n into parts with various restriction on the nature of the parts can be found without any difficulty.

For example, the generating function for the partition of n into distinct (unequal) integral parts is

$$F(x) = (1+x)\,(1+x^2)\,(1+x^3)\,\ldots\ldots = \prod_{m=1}^{\infty} (1+x^m) \tag{2.7}$$

This result can be rewritten as

$$F(x) = \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \ldots\ldots\ldots\ldots$$
$$= \frac{1}{(1-x)\,(1-x^2)\,(1-x^3)\ldots} = \prod_{m=1}^{\infty} (1-x^{2m-1})^{-1} \tag{2.8}$$

Obviously, the right hand side is the generating function for the partition of n into odd integral parts. Thus it follows from (2.7) and (2.8) that the number of partitions of n into unequal parts is equal to the number of its partitions into odd parts. This is indeed a remarkable result.

Another beautiful result follows from Euler's theorem and has the form

$$\frac{(1-x^2)\,(1-x^4)\,(1-x^6)\,\ldots}{(1-x)\,(1-x^3)\,(1-x^5)\,\ldots} = 1+x^1+x^3+x^6+x^{10}+\ldots\ldots \tag{2.9}$$

The powers of x^n are the familiar <u>triangular numbers</u>, $\Delta_n = \frac{1}{2}n(n+1)$ that can be represented geometrically as the number of equidistant points in triangles of different sizes. These points form a triangular lattice. As a generalization of this idea, the <u>square numbers</u> are defined by the number of points in square lattices of increasing size, that is, 1, 4, 9, 16, 25 ....

We next consider the partition function generated by the product $\prod_{m=1}^{\infty} (1-x^m)$ which is the reciprocal of the generating function of the unrestricted partition function p(n) given by (2.2). This product has the representation

$$\prod_{m=1}^{\infty} (1-x^m) = 1+ \sum_{n=1}^{\infty} \{\, p_e(n) - p_o(n)\}x^n = \sum_{n=-\infty}^{\infty} (-1)^n x^{\omega(n)} \tag{2.10}$$

where $p_e(n)$ is the partition of n into an even number of distinct parts, and $p_o(n)$ is the partition of an odd number of distinct parts, and the integers $\omega(n) = \frac{1}{2}(3n^2 - n)$ are called the Euler <u>pentagonal numbers</u> which can be represented geometrically by the number of equidistant points in a pentagon of increasing size.

These points form a pentagonal lattice. Also, it follows from (2.2) and (2.10) by actual computation that

$$F(x) = \frac{1}{\prod_{m=1}^{\infty}(1 - x^m)} = \frac{1}{1-x-x^2+x^5+x^7-x^{12}-x^{15}+\ldots\ldots} \tag{2.11}$$

In view of the fact that the generating functions for the partition of numbers involve infinite products which play an important role in the theory of elliptic and associated functions (Dutta and Debnath, 1965).

Ramanujan made some significant contributions to the theory of partitions. He was not only the first but the only mathematicians who successfully proved several remarkable congruence properties of p(n). Some of his congruences are

$$p(5m + 4) \equiv 0 \pmod 5 \tag{2.12}$$

$$p(7m + 5) \equiv 0 \pmod 7 \tag{2.13}$$

$$p(11m + 5) \equiv 0 \pmod{11} \tag{2.14}$$

All these results are included in his famous conjecture: If $p=5,7$ or 11 and $24n - 1 \equiv 0 \pmod{p^\alpha}$, $\alpha \geq 1$, then

$$p(n) \equiv 0 \pmod{p^\alpha} \tag{2.15}$$

This was a very astonishing conjecture and has led to a good deal of theoretical research and numerical computation on congruence of p(n) using H. Gupta's table (1980) of values of p(n) for $n \leq 300$. However, S. Chowla found that this conjecture is not true for $n = 243$. For this n, $24n-1 = 5831 \equiv \pmod{7^3}$ but

$$p(243) = 133978259344888 \equiv 0 \pmod{7^2}$$

$$\not\equiv 0 \pmod{7^3} \tag{2.16ab}$$

Subsequently, D.H. Lehmer (1936) became deeply involved in the proof of the conjecture and also in the computation of p(n) for large n. G.N. Watson (1938) proved Ramanujan's conjecture for powers of 7. Finally, A.O.L. Atkin (1967) settled the problem by proving the conjecture for powers of 11. Ramanujan's conjecture can now be stated as an important theorem: If $24n-1 \equiv 0 \pmod{d=5^a\,7^b\,11^c}$, then

$$p(n) \equiv 0 \pmod d \tag{2.17}$$

In connection with his famous discovery of several congruence properties, Ramanujan also studied two remarkable partition identities:

$$\sum_{m=0}^{\infty} p(5m + 4)\, x^m = 5\,\frac{\Phi(x^5)^5}{\Phi(x)^6} \tag{2.18}$$

$$\sum_{m=0}^{\infty} p(7m+5)\, x^m = 7\, \frac{\Phi(x^7)^3}{\Phi(x)^4} + 49x\, \frac{\Phi(x^7)^7}{\Phi(x)^8}\,, \tag{2.19}$$

where $$\Phi(x) = \prod_{n=1}^{\infty} (1-x^n)\ . \tag{2.20}$$

The functions on the right side of (2.18) - (2.19) have power series expansions with integer coefficients, Ramanujan's congruences (2.12) - (2.13) follow from these identities. Subsequently, these identities have created a tremendous interest among many researchers including H.B.C. Darling, L.J. Mordell, H. Rademacher and H.S. Zuckermann. They proved Ramanujan's identities by using the theory of modular functions. Proofs without modular functions were given by D. Kruyswijk (1950) and later by O. Kolberg. The method of Kolberg gave not only the Ramanujan identities but many new ones.

Actual computation reveals that the partition function p(n) grows very rapidly with n. D.H. Lehmer (1936) computed p(n) for n = 14,031 to verify a conjecture of Ramanujan which asserts that $p(14{,}031) \equiv 0 \pmod{11^4}$. This assertion was found to be correct. This leads to the question of asymptotic representation of p(n) for large n. During the early part of the 20th century, Hardy and Ramanujan made significant progress in the determination of an asymptotic formula for p(n). Using elementary arguments, they first showed

$$\log p(n) \simeq \left(\frac{2n}{3}\right)^{1/2} + 0\ (n) \qquad \text{as } n \to \infty \tag{2.21}$$

Then, with the aid of a Tauberian Theorem, Hardy and Ramanujan (1918) proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left[\pi \left(\frac{2n}{3}\right)^{\frac{1}{2}}\right] \qquad \text{as } n \to \infty \tag{2.22}$$

This is one of the most remarkable results in the theory of numbers. Equally remarkable was Hardy and Ramanujan's proofs of (2.22). One proof is based on the elementary recurrence relation

$$p(n) = \frac{1}{2} \sum_{k=1}^{\infty} \sigma(k)\, \sigma(n-k),\ p(o) = 1 \tag{2.23}$$

where $\sigma(k)$ is the sum of the divisors of k. The asymptotic approximation of $\sigma(n)$ led to this result (2.22).

Ramanujan and Hardy's second proof was based upon the Cauchy integral formula in complex analysis. In order to outline the proof, we replace real x by a complex z in (2.2) to obtain the Taylor series representation of F(z) and hence the coefficient p(n) of the resulting series can be expressed by the Cauchy integral formula

$$p(n) = \frac{F^{(n)}(o)}{n!} = \frac{1}{2\pi i} \int_C \frac{F(z)dz}{z^{n+1}} \tag{2.24}$$

where F(z) is analytic inside the unit disk $|z| = 1$ in the complex z-plane, C is a suitable closed contour enclosing the origin and lying entirely within the unit disk. It turns out that the unit circle is a natural boundary for F(z).

We choose C as a circle with the origin as center and radius r $(0 < r < 1)$ in the complex z-plane. We make a change of variable $z = \exp(2\pi i\tau)$ in (2.2) so that

$$F(e^{2\pi i\tau}) = \prod_{m=1}^{\infty} (1-e^{2\pi i\tau m})^{-1}$$

$$= \exp\left(\frac{\pi i\tau}{12}\right) [\eta(\tau)]^{-1} \tag{2.25}$$

where $\eta(\tau)$ is the Dedekind η- function which is an analytic function of τ for $\mathrm{Im}(\tau) > 0$.

We now rewrite (2.24) in terms of $\eta(\tau)$ in the form

$$p(n) = \int_{\Gamma} \frac{e^{-2\pi i\lambda_n \tau}\, d\tau}{\eta(\tau)}, \quad \mathrm{Im}(\tau) > 0 \tag{2.26}$$

where Γ is the line segment of length unity, parallel to the real axis, from $-\frac{1}{2} + i\varepsilon$ to $\frac{1}{2} + i\varepsilon$, $\varepsilon > 0$ and $\lambda_n^2 = n - \frac{1}{24}$. We also assume C to be such that Γ is its image under the transformation $z = \exp(2\pi i\tau)$.

The function $\eta(\tau)$ has a simple pole at $\tau=0$. Ramanujan and Hardy proved rigorously that the main contribution to the integral for p(n) given by (2.26) comes from the polar singularity at $\tau=0$ as $n \to \infty$. Thus the asymptotic value of p(n) is given by

$$p(n) \sim \frac{1}{4\lambda_n^2 \sqrt{3}} \exp(\lambda_n K) + O\left(\frac{e^{K\lambda_n}}{\lambda_n^3}\right), \quad n \to \infty, \tag{2.27}$$

where $K = \pi \sqrt{\frac{2}{3}}$.

When λ_n is replaced by n, then (2.27) becomes identical with (2.22).

Finally, H. Rademacher (1937) further improved and fully completed the evaluation of the integral for p(n) by proving an exact formula. He noted that $\eta(\tau)$ has also singular at every point $\tau = \frac{p}{q}$, $(p,q) = 1$ on a segment of the real axis of length unity. He then evaluated contributions to integral (2.26) at its all singular points of the form $\tau = \frac{p}{q}$ and obtained the exact formula

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{q=1}^{\infty} \sqrt{q}\, A_q(n) \frac{d}{dn}\left[\frac{1}{\lambda n} \sinh \frac{K\lambda n}{q}\right], \quad n \geq 1, \tag{2.28}$$

where

$$A_q(n) = \sum_{p \equiv q} \omega_{p,q} \exp(-2\pi i p/q), \quad \omega_{p,q} = \exp[\pi i s(p,q)], \quad (p,q) = 1, \tag{2.29abc}$$

and s(p,q) is the Dedekind sum.

The work of Ramanujan-Hardy's partition function combined with that of Rademacher can be regarded as truely remarkable and have stimulated tremendous interests in subsequent developments in the theory of modular functions. The Ramanujan-Hardy collaboration on the asymptotic analysis for p(n) is one of the monumental results in the history of mathematics and is perhaps best described by J.E. Littlewood (1929) in his review of the collected papers of Srinivasa Ramanujan in the Nature.

3. APPLICATIONS TO STATISTICAL MECHANICS

One of the most remarkable applications of the Ramanujan-Hardy asymptotic formula for p(n) deals with the problems of statistical mechanics. Several authors including Auluck and Kothari (1946), Temperley (1949) and Dutta (1956) discussed the significant role of partition functions in statistical mechanics. The theory of partitions of numbers have been found to be very useful for the study of the Bose-Einstein condensation of a perfect gas. The central problem is the determination of number of ways a given amount of energy can be shared out among different possible states of a thermodynamic assembly. This problem is essentially the same as that of finding the number of partitions of a number into integers under certain restrictions.

We consider a thermodynamic assembly of N non-interacting identical linear simple harmonic oscillators. The energy levels associated with an oscillator are $\varepsilon_m = (m + \frac{1}{2}\hbar\omega$ where m is a non-negative integer, $h=2\pi\hbar$ is the Planck constant and ω is the angular frequency. If E denotes the energy of the assembly, a number n is defined by

$$n(\hbar\omega) = E - \frac{1}{2} N\hbar\omega \tag{3.1}$$

where n denotes (in units of $\hbar\omega$) the energy of the assembly, excluding the residual energy given by the second term of the right side of (3.1).

We denote $\Psi(E)$ for the number of distinct wave functions assigned to the assembly for the energy state E. It is well known that for a Bose-Einstein assembly the number of assigned wave functions is the number of ways of distributing n energy quanta among N identical oscillators without any restriction as to the number of quanta assigned to the oscillator. For a Fermi-Dirac assembly, the energy quanta assigned to all oscillators are all different. For the case of a classical Maxwell-Boltzmann assembly, oscillators are considered as distinguishable from each other, and the number of wave functions is simply the number of ways of distributing n energy quanta among N distinguishable oscillators. This is equal to the number of ways of assigning N elements to n positions, repetitions of any element are permissible.

If $p_d(n)$ denotes the number of partitions of n into exactly d or less than d parts, then $p_d(n) = p(n)$ for $d \geq n$ where p(n) is the number of partitions of n as a sum of positive integers. On the other hand, $q_d(n)$ represents the number of partitions of n into exactly d unequal parts so that $p_d(n) = \sum_{k=1}^{d} q_k(n)$. On the other hand, the number of partitions of n into exactly d or less different parts is denoted by $\tilde{q}_d(n)$, and $Q_d(n)$ stands for the number of partitions of n into exactly d unequal parts so that $\tilde{q}_d(n) = \sum_{k=1}^{d} Q_k(n)$. We also observe the following results:

$$p_d(n) = q_d(n + d) \qquad q_d(n) = Q_d(n + \frac{1}{2} d(d-1)), \tag{3.2ab}$$

$$p_d(n) = Q_d(n + \frac{1}{2} d(d+1)), \; Q_d(n+d) = Q_d(n) + Q_{d-1}(n), \tag{3.3ab}$$

It turns out that

$$\Psi(E) = p_N(n) \qquad \text{for the Bose-Einstein assembly,} \tag{3.4}$$

$$\psi(E) = Q_N(n) + Q_{N-1}(n) = Q_N(n+N) \text{ for the Fermi-Dirac assembly,} \tag{3.5}$$

$$\psi(E) = \frac{{}^N H_n}{N!} = \frac{(N+n-1)\,!}{N!(N-1)!n!} \text{ for the Maxwell-Boltzmann assembly,} \tag{3.6}$$

where N! in (3.6) is inserted to make the entropy expression meaningful. It is important to point out that if $N=O(\sqrt{n}\,)$, both $p_N(n)$ and $Q_N(n+N)$ tend to ${}^N H_n/N!$. This means that for $N << \sqrt{n}$, both the Bose-Einstein statistics and the Fermi-Dirac statistics tend to the classical Maxwell-Boltzmann statistics.

The state function Z for an assembly is defined by

$$Z = \sum_m \psi(E_m)\, e^{-\mu E_m} \tag{3.7}$$

where μ^{-1} is the temperature measured in energy units. For the Bose-Einstein assembly of the linear oscillators, the state function Z is given by

$$Z = \sum_{n=1}^{\infty} p_N(n) \exp[-(n + \tfrac{1}{2} N)\, \mu] \tag{3.8}$$

where $\frac{1}{\mu} = \frac{kT}{\omega\hbar}$, and k is the Boltzmann constant. We can rewrite (3.8) as

$$Ze^{\frac{1}{2}\mu N} = \prod_{r=1}^{N} (1-e^{\mu r})^{-1} \tag{3.9}$$

For the Fermi-Dirac assembly, we have

$$Ze^{\frac{1}{2}\mu N} = \sum_{n=1}^{\infty} [Q_N(n) + Q_{N-1}(n) \exp(-n\mu)$$
$$= \exp\,[-\tfrac{1}{2} N(N-1)\,\mu] \prod_{r=1}^{N} (1-e^{r\mu})^{-1} \tag{3.10}$$

For the classical Maxwell-Boltzmann case, we have

$$Ze^{\frac{1}{2}N\mu} = \frac{1}{N!}(1-e^{-\mu})^{-N} \tag{3.11}$$

It is interesting to point out that as $\mu N^2 \to 0$

$$\prod_{r=1}^{N} (1-e^{-r\mu})^{-1} = (1-e^{-\mu})^{-N} \prod_{r=1}^{N} \{1+e^{-\mu} \dots +e^{-(r-1)\mu}\}^{-1}$$
$$\to \frac{1}{N!}(1-e^{-\mu})^{-N} \tag{3.12}$$

This means that the classical statistics is the limit of both results (3.9) and (3.10).

The above expressions for the state function Z were used to obtain the result for the energy E (or n) and the entropy S. Using the expression for S, asymptotic formulas for the partition functions $p_N(n)$ and p(n) as follows:

$$p_N(n) \sim \frac{1}{N!}\ {}^N H_n = \frac{\exp(2N)\, n^{N-1}}{2\pi\, N^{2N}} \qquad \text{for } N << \sqrt{n} \tag{3.13}$$

$$p_N(n) \sim \frac{1}{4n\sqrt{3}} \exp\left[\pi \sqrt{\frac{2n}{3}} - \frac{1}{\pi}\sqrt{6n}\exp(-\pi N/\sqrt{6n})\right] \text{ for } N >> \sqrt{n} \tag{3.14}$$

Thus, in the limit, $n \to \infty$

$$p_N(n) \to p(n) = \frac{1}{4\sqrt{3}n} \exp\left(\pi \sqrt{\frac{2n}{3}}\right) \tag{3.15}$$

This is the Ramanujan-Hardy asymptotic formula.

A more general result can be derived in the form

$$p_N(n) \sim \frac{\sqrt{D}\exp[\{(\frac{n}{D})^{1/2} + \frac{D}{4}(1-\frac{x}{e^x-1})\} \int_0^x \{\frac{t}{e^t-1} - \log(1-e^{-t})\}\,dt + \frac{x}{2(e^x-1)}}{2\pi n\,[2e(1-e^{-x})\{1-\frac{x^2}{e^x-1}\}]^{1/2}} \tag{3.16}$$

where D is given by

$$D \equiv \int_0^x (e^t-1)^{-1} t\,dt \tag{3.17}$$

and $x=\mu N$ is given by

$$n = \frac{D}{\mu^2} - \frac{1}{2\mu} + \frac{N}{2(e^x-1)} + \frac{1}{24} + \frac{1}{12}\left\{\frac{1}{e^x-1} - \frac{x\,e^x}{(e^x-1)^2}\right\} + O(\mu^2), \tag{3.18}$$

It is important to point out that this general result for $p_N(n)$ reduces to (3.13) as $x=\mu N \to 0$, and to (3.14) as $x=\mu N \to \infty$.

On the other hand, Temperley (1949) applied the Ramanujan-Hardy theory of partitions to discuss results of the Bose-Einstein condensation theory. He considered a problem different from that of Auluck and Kothari. His model consists of N particles obeying Bose-Einstein statistics distributed among infinitely many energy levels $0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots$ of uniform spacing ε in such a way that the total energy is εE. The partition function $p_N(E)$ representing the number of ways of dividing an integer E (energy) into N or less integral parts has the generating function

$$1/(1-x)(1-xz)(1-xz^2)(1-xz^3)\ldots\ldots = \sum_N \sum_E p_n(E)\, x^N z^E, \tag{3.19}$$

Temperley used the Ramanujan-Hardy asymptotic method to compute $p_N(E)$ for large N. His analysis gives

$$p_N(E) \sim \frac{1}{4\sqrt{3}\,E} \exp\left[\pi\frac{\sqrt{2E}}{3} - \frac{\sqrt{6E}}{\pi}\exp\left(-\frac{\pi N}{\sqrt{6E}}\right)\right], \qquad N >> \sqrt{E} \tag{3.20}$$

This result is similar to that of Auluck and Kothari (1946) who considered the problem of distribution of a fixed amount of energy between N harmonic oscillators of equal frequency, and solved it from the statistical mechanics of harmonic oscillators.

Temperley also investigated the Bose-Einstein perfect gas model which is also equivalent to a certain problem in the partition of numbers into sums of squares. The energy levels available to particles in a cubical box of side-length d are given by the expresion

$$\frac{h^2}{8md^2}(p^2+s^2+t^2) \equiv \frac{h^2}{8md^2}K(r,s,t) \tag{3.21}$$

where $K(r,s,t) = (r^2+s^2+t^2)$. Each of these levels may be occupied an integral number of times or not at all. The problem is to find the asymptotic form for the number of distinct partition of an integer E, representing the ratio of the total to the lowest possible energy-separation $h^2/8md^2$, into a sum of the numbers where the order in which the K's are arranged is neglected, but on the other hand, K's like K(1,2,3), K(2,1,3) are distinct from one another, and such K's have to be treated as different, even though they are numerically equal. In all cases, the quantitics r,s,t are positive integers. The upshot of this analysis is the existence of an intermediate temperature region within which the results of earlier theory are unreliable. It is also confirmed the existence of the phenomena of condensation into lowest energy-levels. At the same time, the present investigation gives a transition departure far below 1^o K for a perfect gas of heilium atoms. However, the earlier theory can provide physically sensible results at very high and at very low temperatures.

All these above discussions show a clear evidence for the great importance as well as success of the Ramanujan-Hardy theory of partitions in statistical mechanics.

In an essentially statistical approach to thermodynamic problems, Dutta (1953, 1956) obtained some general results from which different statistics viz., those of Bose, Fermi and Gentile, Maxwell-Boltzmann can be derived by using different partitions of numbers. It is noted that mathematical problems of statisitics of Bose, Fermi, and Gentile are those of partitions of numbers (energy) into partitions in which repetition of parts are restricted differently. In partitions corresponding to Bose statistics any part can be repeated any number of times, that to Gentile statistics any part can be repeated upto d times where d is a fixed positive integers, and that to Fermi statistics no part is allowed to repeat, that is, d=1. All these led to an investigation of a new and different type of partitions of numbers in which repetition of any part is restricted suitably. Motivated by the need of such partition functions and its physical applications to statistical physics, Dutta (1956, 1957) Dutta and Debnath (1959) studied a new partition of number n into any number of parts, in which no part is repeated more than d times. Dutta's partition function is denoted by ${}_dp(n)$. Dutta himself and in collaboration with Debnath proved algebraic and congruence properties of ${}_dp(n)$. The generating function of this partition function is

$$\begin{aligned} f(x) &= \Sigma\, {}_dp(n)\, x^n = \prod_{n=1}^{\infty} (1+x^n+x^{2n}+\dots+x^{dn}) \\ &= \Pi\, \frac{1-x^{n(d+1)}}{1-x^n} = \frac{\Pi\,(1-x^{n(d+1)})}{\Pi\,(1-x^n)} = \frac{\Sigma\, p(n)\, x^n}{\Sigma\, p(n)\, x^{n(d+1)}} \end{aligned} \tag{3.22}$$

where p(n) is the unrestricted partition function due to Hardy and Ramanujan (1918).

He proved the following congruence properties for ${}_dp(n)$:

$$ {}_4p\,(5m+4) \equiv 0 \pmod 5 \tag{3.23}$$

$$ {}_6p\,(7m+5) \equiv 0 \pmod 7 \tag{3.24}$$

$$ {}_{120}p\,(11m+6) \equiv 0 \pmod{11} \tag{3.25}$$

Subsequently, Dutta and Debnath (1957) introduced a new partition function ${}_dp(n|m)$ representing the number of partition of an integer n into m parts with utmost d repetitions. They proved the generating function, congruence properties, recurrence relations and other related properties with examples. Several special cases of this partition functions are discussed with examples. Dutta obtained a simple algebraic formula to calculate successively the numerical values of ${}_dp(n)$ from the values of p(n) and so ultimately from Euler's table. Using a Tauberian Theorem, Dutta proved an asymptotic formula correct up to the exponential order above for ${}_dp(n)$ for large n:

$$ {}_dp(n) \sim \exp\,[\pi\{\tfrac{2}{3}\,n(\tfrac{d}{d+1})\}^{1/2}] \tag{3.26}$$

In particular, for partition of unequal parts (d=1), (3.26) becomes

$$ {}_dp(n) \sim \exp\,[\pi\,(\tfrac{n}{3})^{\frac{1}{2}}] \quad \text{as} \quad n \to \infty \tag{3.27}$$

For unrestricted partitions $(d \to \infty)$, (3.26) reduces to

$$ {}_dp(n) \sim \exp\,[\pi\sqrt{\tfrac{2n}{3}}\,] \qquad \text{for } n \to \infty \tag{3.28}$$

These results are in excellent agreement with those of Hardy and Ramanujan (1918) upto the expotential order. The asymptotic result for the unrestricted partition is found to be very useful for computing the dominant term in the expression for the entropy of the corresponding thermodynamic system.

Dutta's partition function is not only more general than that introduced by earlier authors, but also it is more useful for the study of problems in statistical physics. Mathematical problems of Gentile statistics deals with the partitions of numbers (energy) into parts in which repetition of parts are restricted differently. In partitions corresponding to Bose statistics, any part can be repeated any number of times $(d \to \infty)$. The Fermi statistics deals with the partitions of number into parts in which no part can repeat (d=1). In partitions corresponding to Gentile statistics, any part can be repeated up to d times. In other words, Dutta's partition function ${}_dp(n)$ is found to be useful for an investigation of thermodynamic problems.

As a final example of physical application of Ramanujan-Hardy's theory of partitions of numbers, mention may be made of a paper by Bohn and Kalckar (1937) dealing with calculation of the density of energy levels for a heavy nucleus.

4. CONCLUDING REMARKS

It is hoped that enough has been discussed to give some definite impression of Ramanujan's great character as well as of the range and depth of his contributions to pure mathematics. Throughout his life, Ramanujan was deeply committed to his family and friends. He also expressed an unlimited interest in education and deep compassion for poor students and orphans who needed support for their education. He also profoundly believed in the dignity and work of human being. Ramanujan's entire life was totally dedicated to the pursuit of mathematical truth and dissemination of new mathematical knowledge. His genius was recognized quite early in his life and has never been in question. Indeed, Hardy in his "A Mathematician's Apology" wrote: "I have found it easy to work with others, and have collaborated on a large scale with two exceptional mathematicians (Ramanujan and Littlewood) and this has enabled me to add to mathematics a good deal more than I could reasonably have expected." Also, he said: "All of my best work since then (1911 and 1913) has been bound up with theirs (Ramanujan and Littlewood), ...". There is no doubt at all about Ramanujan's profound and ever-lasting impact on mathematics and mathematical community of the world. Today, one hundred years after his birth, we pay tribute to this great man, and at the same time, we can assess and marvel at the magnitude of his outstanding achievements. By any appraisal, Ramanujan was indeed a noble man and a great mathematician of all time.

5. ACKNOWLEDGEMENTS

The biographical sketch of Srinivasa Ramanujan described in this article is based upon G.H. Hardy's two books entitled Collected Papers of Srinivasa Ramanujan (1927) and Ramanujan (1940). This work was partially supported by the University of Central Florida.

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