ASYMPTOTIC PARTITION FORMULAE

I. PLANE PARTITIONS

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1. The idea of the ordinary partition of an integer has been considered by many writers, with or without restrictions on the size of the parts or on the number of parts. Let us confine our attention for the moment to unrestricted partitions, and regard a partition as involving a definite arrangement of the parts in descending* order of magnitude along a line; thus, a partition of 8 is

$$431. \tag{1.01}$$

Then we may regard this as a one-dimensional or line partition.

This point of view leads us to an obvious generalization to two-dimensional or plane partitions, which may be defined as follows.

Take any line partition of the integer n and arrange the parts in rows and columns, so that a descending* order of magnitude is in evidence in each row from left to right, and in each column from top to bottom. Each row starts from the first column and each column from the first row. Such an arrangement is termed a plane partition of n.

Thus, the particular plane partitions of 8 which are derived from the line partition (1.01) are

$$\left.\begin{array}{cccc} 431 & 43 & 41 & 4 \\ & 1 & 3 & 3 \\ & & & 1 \end{array}\right\} \tag{1.02}$$

The idea of a plane partition was introduced by MacMahon. He deals with the general problem of such partitions, with or without restrictions on one or more of the following:

(i) the size of each part,

(ii) the number of rows,

(iii) the number of columns.

He obtains the generating function† for each of the various cases by means of intricate and beautiful analysis based on his theory of Lattice Functions.

* 'Descending order of magnitude' is used in the wide sense.

† *Combinatory Analysis* (Camb., 1916), vol. ii, pp. 173–243.

The case 'each part not greater than unity' is equivalent to that of unrestricted line partitions. Here the generating function is

$$\prod_{l=1}^{\infty}(1-x^l)^{-1} = 1+\sum_{n=1}^{\infty} p(n)x^n. \tag{1.03}$$

Hardy and Ramanujan* obtained an asymptotic formula for $p(n)$ with an error only $O(n^{-\frac{1}{4}})$.

Of the other cases the unrestricted case is the most difficult and interesting from the asymptotic point of view. For this case the generating function is

$$f(x) = \prod_{l=1}^{\infty}(1-x^l)^{-l} = 1+\sum_{n=1}^{\infty} q(n)x^n, \tag{1.04}$$

where $q(n)$ is the number of plane partitions of n defined as above. In this paper I shall obtain an asymptotic formula for $q(n)$ for large n.

At first sight, the analogy between (1.03) and (1.04) suggests the possibility of some simple proof of (1.04) on the lines of Euler's 'intuitive' proof of (1.03). I have been unable to find any such proof; and as from one of MacMahon's memoirs† it is clear that he devoted much attention to the problem of proving (1.04), it seems unlikely that a simple proof exists.

2.1. I use the following scheme of notation:

$\zeta(s)$ is the zeta-function of Riemann, and B_s is the sth Bernoullian number; that is,

$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s},$$

and

$$B_s = 4s\int_0^{\infty}\frac{y^{2s-1}\,dy}{e^{2\pi y}-1} = \frac{4s\Gamma(2s)\zeta(2s)}{(2\pi)^{2s}}, \tag{2.11}$$

$$A = \zeta(3). \qquad c = 2\int_0^{\infty}\frac{y\log y\,dy}{e^{2\pi y}-1}, \tag{2.12}$$

$$\alpha_s = \frac{B_s B_{s+1}}{4s(s+1)\Gamma(2s+1)} = \frac{2\Gamma(2s+2)\zeta(2s)\zeta(2s+2)}{s(2\pi)^{4s+2}}; \tag{2.13}$$

β_s $(s \leqslant r+1)$ is the coefficient of y^s in the expansion of

$$\exp\left(-\sum_{t=1}^{r+1}\alpha_t y^t\right)$$

* *Proc. London Math. Soc.* (2) 17 (1918), 75–115. The small order of the error in this case is due to special properties of the generating function. The results of the present paper indicate the improbability of anything similar being true in the case of $q(n)$.

† *Phil. Trans. Royal Society*, 211 A, 77.

in ascending powers of y. It is obvious that, so long as $s \leqslant r+1$, β_s is independent of r. Also, if $|y|$ be small,

$$\exp\left(-\sum_{t=1}^{r+1} \alpha_t y^t\right) = \sum_{s=0}^{r+1} \beta_s y^s + O(|y|^{r+2}); \tag{2.14}$$

$b_{s,m}$ is the coefficient of y^{2m} in the expansion of

$$(1+y)^{2s+2m+13/12}(3+2y)^{-m-\frac{1}{2}} \tag{2.15}$$

in ascending powers of y.

H and K are positive numbers whose values vary from one occurrence to another. H is an absolute constant (such as 3), while K depends on r only. By $|u(n)| = \big(Ov(n)\big)$, I mean that, for any fixed r,

$$|u(n)| < Kv(n)$$

for all n; and by $|u(n)| = o\big(v(n)\big)$, I mean that, for any $\epsilon > 0$ and any fixed r,

$$\frac{|u(n)|}{v(n)} < \epsilon$$

for all n greater than some fixed $n_0 = n_0(\epsilon, r)$.

N is a large positive number, to be assigned later, and C_N is the circle

$$|x| = e^{-1/N}$$

in the complex x-plane.

θ is the principal value of arg x,

$$z = \log\frac{1}{x} = \log\left|\frac{1}{x}\right| - i\theta = \rho e^{i\phi},$$

$$w = \Re\left(\frac{\pi}{2z}\right) = \frac{\pi\cos\phi}{2\rho},$$

$$C_N = C_N' + C_N'',$$

where C_N' is that part of C_N on which $|\theta| < 1/N$.

2.2. The object of this paper is to prove the following

THEOREM. *The asymptotic expansion* of $q(n)$ is*

$$\frac{(2^{25}A^7)^{1/36}e^c}{2\pi^{\frac{1}{2}}n^{25/36}}\exp(3\,.\,2^{-\frac{2}{3}}A^{\frac{1}{3}}n^{\frac{2}{3}})\left\{1+\sum_{h=1}^{r}\frac{\gamma_h}{n^{\frac{2}{3}h}}+O(n^{-\frac{2}{3}(r+1)})\right\}, \tag{2.21}$$

where

$$\gamma_h = \frac{1}{\sqrt{\pi}}\left(\frac{4}{A}\right)^{\frac{1}{3}h}\sum_{s=0}^{h}\{(-)^{h-s}\beta_s b_{s,h-s}\Gamma(h-s+\tfrac{1}{2})A^s\}. \tag{2.22}$$

From the product form of $f(x)$ in (1.04), we see that $f(x)$ is an analytic function regular within the unit circle, every point of the

* The numerical values of the constants involved in the first term of the expansion are given at the end of the paper.

circumference being an essential singularity. Then, by Cauchy's theorem, $q(n)$ may be expressed in the form

$$\frac{1}{2\pi i}\int_{C_N}\frac{f(x)\,dx}{x^{n+1}};\tag{2.23}$$

since C_N encloses the origin and lies entirely within the unit circle.

It will appear that, to put it roughly, $x = 1$ is much the 'heaviest' singularity of $f(x)$; and I shall show that, when N is large, the dominant terms in $q(n)$ are supplied by the portion of the integral (2.23) on C'_N. For this purpose, we need two lemmas with regard to the value of $f(x)$ on C'_N, C''_N respectively.

3. Lemma I. *When x is on C'_N, we have*

$$\left|f(x)-e^{c}z^{1/12}\exp\left(\frac{A}{z^2}\right)\left\{\sum_{s=0}^{r+1}\beta_s z^{2s}\right\}\right| < KN^{-2r-4-1/12}e^{AN^2}.\tag{3.01}$$

This is proved by using Cauchy's theorem in a manner which may be adapted to obtain the asymptotic or exact transformation of many similar functions.

We note that for x on C'_N, we have

$$\rho \leqslant \sqrt{2}/N \text{ and } |\phi| < \tfrac{1}{4}\pi.$$

We understand by E a number whose absolute value is less than

$$Ke^{-H/\rho}.$$

It is obvious that, since $\cos\phi > 1/\sqrt{2}$,

$$e^{-(H/\rho)\cos\phi} < e^{-H/\rho} = E,$$

for x on C'_N. Also

$$\rho^{\alpha}(\log\rho)^{\beta}e^{-H/\rho} = E,$$

if α and β are numbers (positive or negative) depending only on r.

3.1. Suppose that p is a positive integer and δ a small positive number. Let Γ_p be a contour in the t-plane running from $t=\delta$ to $t=p+\frac{1}{2}$, coincident with the real axis except near $t=1, 2,\ldots, p$, where we pass round small semicircles above these points. Let Δ be the quarter-circle with centre at the origin running from $i\delta$ to δ. Finally, let Γ'_p, Δ' be the reflections of Γ_p, Δ in the real axis.

We take that value of $\log(1-e^{-tz})$ which tends to zero as $t \to +\infty$ along the real axis, and note that the only singularities of

$$\psi(t) = \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}},\tag{3.11}$$

apart from the origin, are simple poles at $t = \pm l$, and logarithmic singularities at $t = \pm 2l\pi i/z$, where l runs through all positive integral values.

We draw two straight lines, one through $t = iw$, parallel to the real axis, and the other through $t = p+\frac{1}{2}$, parallel to the imaginary axis. These two straight lines form an indented rectangle with Δ, Γ_p, and the imaginary axis from $i\delta$ to iw. By Cauchy's theorem, since the integrand has no singularities within the rectangle,

$$\int_{\Gamma_p} + \int_{iw}^{i\delta} \psi(t)\,dt = -\int_{\Delta} + \int_{iw}^{p+\frac{1}{2}+iw} + \int_{p+\frac{1}{2}+iw}^{p+\frac{1}{2}} \psi(t)\,dt = I_1 + I_2 + I_3. \tag{3.12}$$

The reader will readily find that, for $|\phi| < \frac{1}{4}\pi$,

$$|I_1| < H(\delta \log \delta + \delta |\log z|),$$

and
$$|I_3| < H\left(p + \frac{\pi}{2\rho}\right) e^{-p\rho/\sqrt{2}}.$$

Now let δ tend to zero and p to infinity. The limits of integration in I_2 are now iw and $iw+\infty$; and Γ_p becomes $\Gamma_\infty = \Gamma$ (say). The equation (3.12) becomes

$$\int_{\Gamma} \psi(t)\,dt + \int_{iw}^{0} \psi(t)\,dt = I_2. \tag{3.13}$$

Let us now consider I_2. Making the transformation $v = tz$, we find that

$$|I_2| < He^{-(\pi^2/\rho)\cos\phi} \int_{iw}^{\infty+iw} |t \log(1-e^{-tz})|\,|dt| = EI(\phi), \tag{3.14}$$

where
$$I(\phi) = \int_{L} |v|\,|\log(1-e^{-v})|\,|dv|,$$

and L is the half straight line in the v-plane from

$$v = \tfrac{1}{2}\pi i e^{i\phi} \cos\phi$$

inclined at an angle ϕ to the positive direction of the real axis. Divide L into L_1 and L_2 so that, on L, $\Re(v) \leqslant \frac{1}{2}\pi$. Then L_1 is of bounded length, and the integrand on L_1 is uniformly bounded for $|\phi| \leqslant \frac{1}{4}\pi$. Also, on L_2, $\Re(v) > H|v|$, and so we have

$$\int_{L_2} |v|\,|\log(1-e^{-v})|\,|dv| < H \int_{L_2} |ve^{-v}|\,|dv| < H \int_{L_2} |v|\,e^{-H|v|}\,|dv| < H.$$

Hence $I(\phi) < H$, and we have $I_2 = E$.

(3.13) now becomes

$$\int_{\Gamma} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\,dt = \int_0^{iw} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\,dt + E. \tag{3.15}$$

By similar reasoning, we have also

$$\int_{\Gamma'} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}\,dt = \int_0^{-iw} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}\,dt + E. \tag{3.16}$$

By Cauchy's theorem we know that

$$\int_{\Gamma'} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}\,dt - \int_{\Gamma} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\,dt$$

$$= \sum_{l=1}^{\infty} l\log(1-e^{-lz}) + \int_{\Gamma} \left(\frac{t\log(1-e^{-tz})}{e^{2\pi it}-1} - \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\right) dt$$

$$= -\log f(x) - \int_0^{\infty} t\log(1-e^{-tz})\,dt$$

$$= -\log f(x) + \frac{A}{z^2}. \tag{3.17}$$

Finally, from (3.15), (3.16), and (3.17), we have

$$\log f(x) = \frac{A}{z^2} + \int_0^{iw} \frac{t\log(1-e^{-tz})}{1-e^{-2\pi it}}\,dt - \int_0^{-iw} \frac{t\log(1-e^{-tz})}{e^{2\pi it}-1}\,dt + E,$$

which reduces to

$$\log f(x) = \frac{A}{z^2} + 2\int_0^{w} \frac{y\log(2\sin\frac{1}{2}yz)}{e^{2\pi y}-1}\,dy + E. \tag{3.18}$$

3.2. We now proceed to calculate the value of

$$2\int_0^{w} \frac{y\log(2\sin\frac{1}{2}yz)}{e^{2\pi y}-1}\,dy. \tag{3.21}$$

From the well-known formula

$$\sin\tau = \tau\prod_{m=1}^{\infty}\left(1-\frac{\tau^2}{m^2\pi^2}\right),$$

we have at once, for $|\tau| < \pi$,

$$\log\sin\tau = \log\tau - \sum_{s=1}^{\infty}\sum_{m=1}^{\infty} \frac{\tau^{2s}}{sm^{2s}\pi^{2s}} = \log\tau - \sum_{s=1}^{\infty} \frac{\zeta(2s)\tau^{2s}}{s\pi^{2s}}. \tag{3.22}$$

If we now write $\tau = \frac{1}{2}yz$, we have, on the range of integration in (3.21),

$$|\tau| = |\tfrac{1}{2}yz| \leqslant \tfrac{1}{2}w|z| \leqslant \tfrac{1}{4}\pi.$$

The series (3.22) is uniformly convergent for such values of τ, and so we may write

$$2\int_0^w \frac{y\log(2\sin\frac{1}{2}yz)}{e^{2\pi y}-1}\,dy$$
$$= 2\int_0^w \frac{y\log(yz)\,dy}{e^{2\pi y}-1} - 2\left(\sum_{s=1}^{r+1} + \sum_{s=r+2}^{\infty}\right)\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^w \frac{y^{2s+1}\,dy}{e^{2\pi y}-1}$$
$$= I_4 + S_1 + S_2. \tag{3.23}$$

Now we have

$$I_4 = 2\int_0^\infty \frac{y\log(yz)}{e^{2\pi y}-1}\,dy + O(\rho^{-1}e^{-(\pi^2/\rho)\cos\phi}\log\rho)$$
$$= 2\int_0^\infty \frac{y\log y}{e^{2\pi y}-1}\,dy + 2\log z\int_0^\infty \frac{y\,dy}{e^{2\pi y}-1} + E$$
$$= c + \tfrac{1}{12}\log z + E, \tag{3.24}$$

since
$$\int_0^\infty \frac{y\,dy}{e^{2\pi y}-1} = \tfrac{1}{4}B_1 = \tfrac{1}{24}.$$

Also, by (2.11) and (2.13),

$$-S_1 = 2\sum_{s=1}^{r+1}\frac{\zeta(2s)z^{2s}}{s(2\pi)^{2s}}\int_0^\infty \frac{y^{2s+1}\,dy}{e^{2\pi y}-1} + E$$
$$= \sum_{s=1}^{r+1}\alpha_s z^{2s} + E. \tag{3.25}$$

Finally,

$$|S_2| < K\rho^{2r+4}\int_0^w \frac{y^{2r+5}}{e^{2\pi y}-1}\left\{\sum_{s=0}^{\infty}\left(\frac{y\rho}{2\pi}\right)^{2s}\right\}dy$$
$$= K\rho^{2r+4}\int_0^w \frac{y^{2r+5}\,dy}{\left\{1-\left(\frac{y\rho}{2\pi}\right)^2\right\}(e^{2\pi y}-1)}$$
$$< K\rho^{2r+4}\int_0^\infty \frac{y^{2r+5}\,dy}{e^{2\pi y}-1} < K\rho^{2r+4}. \tag{3.26}$$

Combining (3.18), (3.24), (3.25), and (3.26), we see that the error terms may all be included in one term, since

$$E < Ke^{-H/\rho} < K\rho^{2r+4}.$$

Hence,

$$\log f(x) = \frac{A}{z^2} + \frac{1}{12}\log z + c - \sum_{s=0}^{r+1} \alpha_s z^{2s} + O(\rho^{2r+4});$$

and from this we have

$$f(x) = e^c z^{1/12} e^{A/z^2} \left\{ \sum_{s=0}^{r+1} \beta_s z^{2s} + O(\rho^{2r+4}) \right\}.$$

Now, on C'_N, $\rho \leqslant \sqrt{2}/N$, and

$$\left| \exp\left(\frac{A}{z^2}\right) \right| \leqslant \exp\left|\frac{A}{z^2}\right| \leqslant e^{AN^2};$$

so we have

$$\left| f(x) - e^c z^{1/12} e^{A/z^2} \left\{ \sum_{s=0}^{r+1} \beta_s z^{2s} \right\} \right| < KN^{-2r-4-1/12} e^{AN^2},$$

the result of Lemma I.

4. Lemma II. *Given any $\epsilon > 0$, we can find an $N_0 = N_0(\epsilon)$, such that, for all $N > N_0$ and all x on C''_N, we have*

$$|f(x)| < \exp\{(A - \tfrac{1}{2} + \epsilon)N^2\}.$$

The proof of this is very simple. We have

$$\log f(x) = -\sum_{l=1}^{\infty} l \log(1 - x^l) = \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \frac{l x^{ml}}{m} = \sum_{m=1}^{\infty} \frac{x^m}{m(1-x^m)^2};$$

and so

$$|\log f(x)| \leqslant \frac{|x|}{|1-x|^2} + \sum_{m=2}^{\infty} \frac{|x|^m}{m(1-|x^m|)^2}$$

$$= \sum_{m=1}^{\infty} \frac{|x|^m}{m(1-|x^m|)^2} - \left\{ \frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} \right\}.$$

But
$$\sum_{m=1}^{\infty} \frac{|x|^m}{m(1-|x^m|)^2} = \log f(|x|) = AN^2 + o(N^2),$$

by Lemma I. Also

$$\frac{|x|}{(1-|x|)^2} - \frac{|x|}{|1-x|^2} = \frac{|x|}{(1-|x|)^2}\left\{1 - \left(\frac{1-|x|}{|1-x|}\right)^2\right\} \geqslant \tfrac{1}{2}N^2 + o(N^2),$$

since
$$\frac{1-|x|}{|1-x|} \leqslant \frac{1}{\sqrt{2}} + O\left(\frac{1}{N}\right) \qquad \text{for } |\theta| \geqslant \frac{1}{N}.$$

So we have finally

$$|\log f(x)| \leqslant (A - \tfrac{1}{2})N^2 + o(N^2) < (A - \tfrac{1}{2} + \epsilon)N^2,$$

if $N > N_0(\epsilon)$.

5.1. I now put $N=(n/2A)^{\frac{1}{3}}$, so that $n=2AN^3$. It is convenient to work mainly in terms of N, rather than n. We have

$$q(n)=\frac{1}{2\pi i}\int\limits_{C_N}\frac{f(x)\,dx}{x^{n+1}}$$

$$=\frac{1}{2\pi i}\Bigl(\int\limits_{C'_N}+\int\limits_{C''_N}\Bigr)=J_1+J_2.$$

By Lemma I,

$$J_1=\frac{1}{2\pi i}\int\limits_{(1-i)/N}^{(1+i)/N} f(e^{-z})e^{2AN^3z}\,dz$$

$$=\frac{e^c}{2\pi i}\int\limits_{(1-i)/N}^{(1+i)/N}\Bigl(\sum_{s=0}^{r+1}\beta_s z^{2s+1/12}\Bigr)\exp\Bigl(\frac{A}{z^2}+2AN^3z\Bigr)\,dz+$$

$$+O(N^{-2r-5-1/12}e^{3AN^2})$$

$$=e^c\sum_{s=0}^{r+1}\frac{\beta_s P_s}{N^{2s+1+1/12}}+O(N^{-2r-5-1/12}e^{3AN^2}),$$

where $$P_s=\frac{N^{2s+1+1/12}}{2\pi i}\int\limits_{(1-i)/N}^{(1+i)/N} z^{2s+1/12}\exp\Bigl(\frac{A}{z^2}+2AN^3z\Bigr)\,dz$$

$$=\frac{1}{2\pi i}\int\limits_{1-i}^{1+i} v^{2s+1/12}\exp\Bigl\{AN^2\Bigl(2v+\frac{1}{v^2}\Bigr)\Bigr\}\,dv,$$

if we put $v=Nz$.

5.2. Our next step is to obtain an asymptotic expansion of P_s in descending powers of N^2 by the method of 'steepest descents'.*

The curve of steepest descent through $v=1$ is found by putting

$$\mathfrak{I}(2v+v^{-2})=0.$$

If we write $v=X+iY$, and reject the factor Y, we find the equation of the curve $\mathfrak{C}$, namely,

$$(X^2+Y^2)^2=X. \tag{5.21}$$

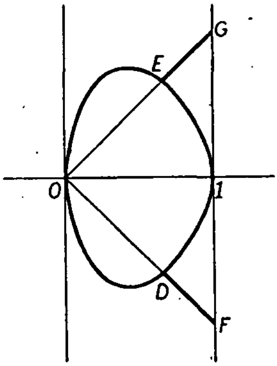
This is the equation of a closed curve touching $X=1$ at $(1,0)$, and $X=0$ at $(0,0)$, and cutting $X=-Y$, $X=Y$ at the points $D\,(2^{-\frac{2}{3}},-2^{-\frac{2}{3}})$, $E\,(2^{-\frac{2}{3}},2^{-\frac{2}{3}})$ respectively. $\mathfrak{C}$ lies entirely in the strip $0\leqslant X\leqslant 1$. The figure shows the general shape of the curve.

* Watson, *Proc. London Math. Soc.* (2) 17 (1918), 117, gives references to memoirs discussing this method.

We consider the v-plane as cut from 0 to $-\infty$ along the real axis, and we take that value of $v^{1/12}$ which is real and positive at $v=1$. We regard the contour $\mathfrak{C}$ as starting from 0 and being described in the counter-clockwise sense.

Let us write
$$\xi_s(v)=\frac{v^{2s+1/12}}{2\pi i}\exp\left\{AN^2\left(2v+\frac{1}{v^2}\right)\right\},$$
and
$$\bar{P}_s=\int_{\mathfrak{C}}\xi_s(v)\,dv.$$

If F, G are the points $1-i$, $1+i$, we have
$$\Re(v^{-2})=\frac{X^2-Y^2}{(X^2+Y^2)^2}\leqslant 0$$



on the straight lines DF, EG and the parts OD, EO of $\mathfrak{C}$. Hence, on these contours, since $X<1$, $|v|$ is bounded, and $s\leqslant r+1$, we have
$$|\xi_s(v)|=\frac{|v|^{2s+1/12}}{2\pi}e^{2AN^2X}\leqslant Ke^{2AN^2}.$$
It follows that
$$|P_s-\bar{P}_s|\leqslant\left|\int_D^F\xi_s(v)\,dv\right|+\left|\int_G^E\right|+\left|\int_E^O\right|+\left|\int_O^F\right|<Ke^{2AN^2}. \tag{5.22}$$

In $\bar{P}_s$, we now put
$$t=-i\left(\frac{v-1}{v}\right)(2v+1)^{\frac{1}{2}}, \tag{5.23}$$
that value of the square root being taken which is positive at $v=1$. Then
$$t^2=3-2v-v^{-2},$$
and $\mathfrak{C}$ transforms into the real axis in the t-plane. So we have
$$\bar{P}_s=e^{3AN^2}\int_{-\infty}^{+\infty}\chi_s(t)e^{-AN^2t^2}\,dt,$$

where $$\chi_s(t) = \frac{v^{2s+1/12}}{2\pi i}\frac{dv}{dt}.$$

5.3.* Near $v = 1$, we can expand (5.23) in a power series

$$t = \sum_{m=1}^{\infty} d_m (v-1)^m,$$

where $d_1 = -i\sqrt{3}$. Since $d_1 \neq 0$, by reversion of the series we have

$$v = 1 + \sum_{1}^{\infty} g_m t^m,$$

and $$\frac{dv}{dt} = \sum_{1}^{\infty} m g_m t^m,$$

each with a positive radius of convergence. Hence we have

$$\chi_s(t) = \sum_{m=0}^{\infty} a_m t^m, \tag{5.31}$$

the series having a radius of convergence greater than some K.

Since $$t^2 = 3 - 2v - v^{-2},$$

we see that $$t\frac{dt}{dv} = \frac{1-v^3}{v^3};$$

and so $$\frac{dv}{dt} = \frac{v^3 t}{1-v^3} = \frac{iv^2(2v+1)^{\frac{1}{2}}}{1+v+v^2}.$$

Then, on $\mathfrak{C}$, $$\left|\frac{v^{2s+1/12}}{2\pi i}\frac{dv}{dt}\right| < K;$$

and $|\chi_s(t)|$ is therefore bounded on the whole of the real axis in the t-plane.

Since (5.31) holds on an interval of the real axis near $t = 0$, we have, on the whole of the real axis,

$$\left|\chi_s(t) - \sum_{m=0}^{2r+3} a_m t^m\right| < K|t|^{2r+4}.$$

Hence $$\bar{P}_s = \int_{-\infty}^{+\infty} \chi_s(t) e^{-AN^2t^2}\, dt$$

$$= \sum_{m=0}^{2r+3} \int_{-\infty}^{+\infty} a_m t^m e^{-AN^2t^2}\, dt + M_r,$$

where $$|M_r| < K\int_{-\infty}^{+\infty} t^{2r+4} e^{-AN^2t^2}\, dt < \frac{K}{N^{2r+5}}.$$

* I am indebted to Watson's paper (loc. cit., pp. 133, 137) for the method I use in this paragraph.

Now
$$\int_{-\infty}^{\infty} t^m e^{-AN^2t^2}\,dt = 0.$$
if m is odd; while if m is even,
$$\int_{-\infty}^{\infty} t^m e^{-AN^2t^2}\,dt = \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})}{(AN^2)^{\frac{1}{2}(m+1)}}.$$
So finally we have the asymptotic expansion
$$P_s = \overline{P}_s + O(e^{2AN^2})$$
$$= e^{3AN^2}\sum_{m=0}^{r+1}\frac{a_{2m}\Gamma(m+\frac{1}{2})}{(AN^2)^{m+\frac{1}{2}}} + O\left(\frac{e^{3AN^2}}{N^{2r+5}}\right). \tag{5.32}$$

5.4. To calculate a_{2m} we proceed as follows. Near $t=0$,
$$\chi_s(t) = \sum_{m=0}^{\infty} a_m t^m,$$
and so
$$a_{2m} = \frac{1}{2\pi i}\int^{(0+)} \chi_s(t)\frac{dt}{t^{2m+1}},$$
the integral being taken along a small loop around $t=0$.

Then
$$a_{2m} = -\frac{1}{4\pi^2}\int^{(1+)} \frac{v^{2s+1/12}}{t^{2m+1}}\,dv$$
$$= \frac{(-1)^m}{4\pi^2 i}\int^{(1+)} \frac{v^{2s+1/12+2m+1}}{(v-1)^{2m+1}(2v+1)^{m+\frac{1}{2}}}\,dv.$$
Putting $v = 1+y$, we have
$$a_{2m} = \frac{(-1)^m}{4\pi^2 i}\int^{(0+)} \frac{(1+y)^{2s+2m+1+1/12}\,dy}{y^{2m+1}(3+2y)^{m+\frac{1}{2}}}$$
$$= \frac{(-1)^m b_{s,m}}{2\pi}, \qquad \text{by (2.15).}$$

5.5. By Lemma II,
$$|J_2| < \exp\{(A-\tfrac{1}{2}+\epsilon)N^2+nN^{-1}\}$$
$$= \exp\{(3A-\tfrac{1}{2}+\epsilon)N^2\} < K(N^{-2r-5-1/12}e^{3AN^2}),$$
if we take $\epsilon < \frac{1}{2}$. Combining this with (5.32), we have
$$q(n) = J_1+J_2 = e^c\sum_{s=0}^{r+1}\frac{\beta_s P_s}{N^{2s+13/12}} + O\left(\frac{e^{3AN^2}}{N^{2r+5+1/12}}\right)$$
$$= \frac{e^c}{2\pi}e^{3AN^2}\sum_{s=0}^{r+1}\sum_{m=0}^{r+1}\frac{(-1)^m\beta_s b_{s,m}\Gamma(m+\frac{1}{2})}{A^{m+\frac{1}{2}}N^{2(m+s+1)+1/12}} +$$
$$+O(N^{-2r-5-1/12}e^{3AN^2}).$$

Let us now put $h = m+s$, and sum with respect to h and s. Then s runs from 0 to h; and we may collect all terms for which $h > r$ in one error term. We have

$$q(n) = \frac{e^c e^{3AN^2}}{2\pi A^{\frac{1}{2}} N^{25/12}} \left\{ \sum_{h=0}^{r} \left(\frac{A}{4}\right)^{\frac{1}{2}h} \frac{\pi^{\frac{1}{2}} \gamma_h}{A^h N^{2h}} \right\} + O\left(\frac{e^{3AN^2}}{N^{2r+49/12}} \right).$$

If we now write $N = (n/2A)^{\frac{1}{3}}$, we have the result of the theorem.

6. It is clear that the method I have used could be extended to determine the asymptotic expansion of the nth coefficient of other functions of a similar type. The essential step in each case is the proof of two lemmas corresponding to those proved here.

The next paper of this series will be concerned with the different idea of 'weighted' partitions, and, in particular, with the coefficients of the function

$$\prod_{l=1}^{\infty} (1-ax^l)^{-1}.$$

The case $a > 1$ seems of the greater interest, but the appropriate method is entirely distinct from the one used here. The case $a < 1$, however, which is of less interest in itself, provides a simple example of an application of the method of this paper.

Note. The numerical evaluation of the number c involves that of the series

$$\sum_{m=2}^{\infty} \frac{\log m}{m^2} = 0{\cdot}9375482\ldots.$$

We know* that $$A = \zeta(3) = 1{\cdot}20205690\ldots,$$

and so we have $$\frac{(2^{25}A^7)^{1/36} e^c}{2\pi^{\frac{1}{2}}} = 0{\cdot}4009989\ldots.$$

We have also, for the coefficient of $n^{\frac{2}{3}}$ in the index of the exponent, the value

$$3{\cdot}2^{-\frac{2}{3}} A^{\frac{1}{3}} = 2{\cdot}0094\ldots.$$

The values of $q(n)$ for $n = 1, 2, \ldots, 16$ have been calculated by Macmahon, and appear in the right-hand column of the table on p. 332, vol. ii, of his *Combinatory Analysis*. In this table, $q(n)$ appears as the number of 'solid graphs' of n nodes, but the two definitions are equivalent.

* Knopp, *Infinite Series*, p. 561.