

Self-Organized Criticality in the Olami-Feder-Christensen Model

Josué X. de Carvalho* and Carmen P.C. Prado†

Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970, São Paulo, SP, Brazil

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A system is in a self-organized critical state if the distribution of some measured events obeys a power law. The finite-size scaling of this distribution with the lattice size is usually enough to assume that the system displays self-organized criticality. This approach, however, can be misleading. In this paper we analyze the behavior of the branching rate σ of the events to establish whether a system is in a critical state. We apply this method to the Olami-Feder-Christensen model to obtain evidence that, in contrast to previous results, the model is critical in the conservative regime only.

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In spite of many efforts and more than a decade of studies, the presence of self-organized critical behavior in nature (and in some computer models) is a matter of controversy. The concept of self-organized criticality (SOC) was originally proposed by Bak, Tang, and Wiesenfeld to describe the appearance of scale invariance in nature. The idea was presented through the study of the behavior of avalanches in a sandpile “toy” model [1]. This simple model displayed the fundamental properties associated with self-organized criticality. Under a slow driven perturbation the system evolves to a critical state, with no characteristic time and length scales. Once in this state, the response of the system to the slow perturbation has no typical length, and even a small perturbation (as the addition of a single grain of sand) can start a big avalanche.

Avalanching behavior as well as scale invariance have been experimentally observed in a variety of situations in nature, ranging from such different phenomena as earthquakes [2] or magnetic systems (the Barkhausen effect) [3], to biological problems such as the evolution of species [4] or lung inflation [5], just to give some examples. Although by now the initial attempt to explain the appearance of all linear scaling in nature through the concept of SOC may seem a little naive, the ubiquity of its presence is still a strong suggestion that some kind of “robust” and general mechanism may be behind many of these phenomena. The concept of SOC has become polemic, and, up to now, there is no general agreement about the ingredients necessary to create the self-organized critical state. Particularly, there are discussions about the need for some type of local conservation as an essential ingredient of the system to display SOC. The existence of SOC in nonconservative models would be highly desirable in this context, since, in practice, some type of dissipation is always present in nature.

One of the best successful applications of the ideas of SOC for nonconservative systems is the investigations of Olami, Feder, and Christensen on a model for the dynamic of earthquakes (hereafter called OFC model [6]). In this model there is a parameter α that controls the level of conservation. Based on strong numerical evidence [7], it

has been taken as an example of a system that has self-organized criticality in the nonconservative regime, that is, for $\alpha < 0.25$.

In this paper we revisited the OFC model, but with a different technique. Instead of looking for power laws in distribution functions of avalanche sizes versus lattice sizes, we looked at the behavior of the average branching rate, both in the conservative and in the nonconservative regimes. In contrast to previous evidence, we concluded that the OFC model is critical only for $\alpha \approx 0.25$ (that is, in the conservative regime). For values of α close to but smaller than 0.25, this model could be classified as “almost critical.” This means that, although being critical only for $\alpha = 0.25$, for all practical purposes the system behaves as if it were critical for a wide range of values of α , with well-defined power laws over many decades.

In a recent paper, Kinouchi and Prado [8] showed that some models which exhibit numerical evidence of self-organized criticality in a wide range of the coupling parameters were indeed what they called “almost critical.” Through the analysis of the branching rate σ as a function of the dissipation parameter α , they have shown that, although those systems are critical only for $\alpha = \alpha_c$, there is a rather large region around this point where approximate scale invariance holds. They called this behavior “almost critical” since, in practice, it can hardly be distinguished from “true” criticality based on the usual numerical evidences only. By usual numerical evidences we mean power-law behavior and scale invariance in distribution functions (the avalanche size distribution function, for instance). They also suggested that the analysis of the branching rate σ (where $0 \leq \sigma \leq 1$) as a function of the coupling constant α could be a more efficient way to determine whether a model is critical or not. Looking for power laws in lattices of increasing sizes is not a very efficient way to determine if a system is in fact critical, and this approach has already led to mistakes [9]. If the analyzed lattices are not big enough, the distribution functions of avalanche sizes $F(s)$ are power laws, even if the model does not display SOC. Because the computational cost of simulating the OFC model (and many others) in big

lattices is prohibitive and there is no way to know, beforehand, if the considered lattices are big enough to show the real characteristics of the dynamical behavior of the system, such an approach is hardly conclusive.

It has been shown that some SOC models, with no spatial correlations and in the thermodynamic limit, can be mapped into a branching process [10]. A branching process is a Markovian process and can be characterized by a sequence of random variables $\{P(n)\}_{n=0}^{\infty}$, $n \in N$, where $P(n)$ represents the total number of individuals of the n th generation. Consider a group of individuals (ancestors) that can replicate, giving birth to some descendants, and let p_i ($i = 0, 1, \dots, \infty$) be the probability of an ancestor to give birth to i descendants. Each of its descendants in turn can give birth to other descendants with the same probability p_i so that p_i does not depend on the previous generations or on the number of descendants of other individuals in the same generation. The branching rate, $\sigma = \sum_{i=0}^{\infty} i p_i$, is then defined as the average number of descendants per ancestor. It is a well-known result that, in order to have a critical branching process, one must have $\sigma = 1$. Then the total number of descendants $P(n)$ in each generation (the size of the “colony”) behaves as a power law $P(n) \propto n^{-3/2}$ [11]. On the basis of these considerations about the branching rate, and using different approaches, several authors [12] were able to show that the random version of the OFC model was critical in the conservative regime only.

Therefore, we decided to use this same approach to revisit the original Olami-Feder-Christensen model [6]. This coupled-map lattice model is inspired on the spring block model developed by Burridge and Knopoff [13]. Each site (i, j) of a square lattice is associated with a continuous “energy” F_{ij} , initially set to a random value in the interval $(0, F_c)$. The system is driven by a global perturbation that increases the energy of all sites uniformly and simultaneously. This process goes on until eventually one site becomes supercritical, that is, $F_{ij} \geq F_c$. This site becomes unstable and the system then relaxes according to the rules

$$F_{ij} \rightarrow 0$$

and

$$F_{nn} \rightarrow F_{nn} + \alpha F_{ij},$$

where F_{nn} are the four nearest neighbors of site (i, j) . The parameter α controls the level of conservation of the model. If $\alpha = 0.25$, the system is said to be “conservative,” that is, all of the energy (or strength) lost by the site (i, j) is distributed to its neighbors. This relaxation rule can possibly produce a chain reaction that ends only when all sites are stable again ($F_{ij} < F_c, \forall i, j$). As in the original work, we assume open boundaries. Also, as shown in Ref. [6], one must have $\alpha < 0.25$ to mimic the dynamic of a real earthquake (some energy or “strength” is always lost to the upper moving tectonic plate). This model is believed to display self-organized criticality even when the dynamic is nonconservative ($0 < \alpha < 0.25$). This is a result not yet fully understood, and the value of the lower bound for α (if it exists), under which the system has a localized behavior, has been a matter of controversy (note that we know that $\alpha = 0 \Rightarrow \sigma = 0$, and, for $\alpha = 0.25$, we should have $\sigma = 1$). Because it is a model defined on a lattice, analytical approaches are difficult and most of the results have been obtained from computer simulations.

As the existence of a lattice introduces spatial correlations, it is not possible to define the probability p_i analytically. We estimate the branching rate σ numerically [$\sigma = \langle n_d \rangle$, where $\langle n_d \rangle$ is the average number of supercritical sites (descendants) originated by an unstable site]. Just for comparison, we also study the random neighbor version of the OFC (R-OFC) model [14], for which there are some analytical results [12] showing that the model is critical for $\alpha = 0.25$ only.

Our results are presented in Tables I and II and in Figs. 1–3. We checked the dependence of σ on the lattice size (see Figs. 2 and 3), and special care has been taken to guarantee that the long transients were eliminated. We also checked the effects of the boundaries. In the OFC model we considered open boundaries to calculate σ , taking into account that the average number of descendants for a boundary site is the number of unstable sites it gives birth to, divided by the real number of neighbors of the “ancestor” site (three for a border site and two for a corner site). The R-OFC model was simulated without borders. In most of the cases, we first generated different stationary configurations from different random initial configurations. The errors were estimated by averaging

TABLE I. Values of σ , σ_b , and $\langle s \rangle$ for different values of the conservative parameter α in the Olami-Feder-Christensen model. Results presented are those obtained with the biggest lattice (L_{\max}) we were able to simulate. They represent the average of results obtained for different initial configurations, and the errors are the errors associated with those averages.

α	σ	σ_b	$\langle s \rangle$	L_{\max}
0.15	0.7052 ± 0.0002	0.7151 ± 0.0002	3.40 ± 0.02	100
0.18	0.8361 ± 0.0003	0.8430 ± 0.0003	6.08 ± 0.08	150
0.21	0.9125 ± 0.0002	0.9205 ± 0.0002	11.0 ± 0.6	100
0.22	0.9546 ± 0.0009	0.9581 ± 0.0009	21.4 ± 0.4	200
0.23	0.982 ± 0.001	0.983 ± 0.001	53 ± 3	400
0.24	0.9938 ± 0.0004	0.9946 ± 0.0004	148 ± 9	400
0.25	1.000003 ± 0.000009	1.000068 ± 0.000009	39839 ± 68	400

TABLE II. Values of σ , σ_b , and $\langle s \rangle$ for different values of the conservative parameter α in the random version of the Olami-Feder-Christensen model. Results presented are those obtained with the biggest lattice (L_{\max}) we were able to simulate. They represent the average of results obtained for different initial configurations, and the errors are the errors associated with those averages.

α	σ	$\langle s \rangle$	L_{\max}
0.15	0.6006 ± 0.0003	2.083 ± 0.001	100
0.18	0.7140 ± 0.0003	3.497 ± 0.004	100
0.21	0.8595 ± 0.0002	7.12 ± 0.01	400
0.22	0.9297 ± 0.0002	14.22 ± 0.04	500
0.23	0.9876 ± 0.0002	81 ± 1	800
0.24	0.99923 ± 0.00008	1306 ± 80	1000

results obtained for different initial configurations of the lattice (the errors so obtained are usually bigger than the ones obtained by averaging σ during many generations, except when the system is conservative). The number of iterations needed to reach the stationary state is very big, and grows with the lattice size. In the OFC model the transient is bigger for smaller values of α , while in the R-OFC model the transient grows as α grows, making it impossible to simulate the case $\alpha = 0.25$ (the point in the graph in this case was obtained from theoretical results).

Once we were sure to have a stationary configuration, we analyzed the statistics of 100 000–5 000 000 avalanches in the stationary state, in order to obtain (a) the average avalanche size $\langle s \rangle$, (b) the branching rate σ (weighting border sites), (c) the branching rate in the bulk σ_b (taking into account only sites in the bulk), and (d) the average number of generations in an avalanching process $\langle n \rangle$. Table I shows the results for the OFC model and Table II shows the results for the R-OFC model. Note that, in the model R-OFC, $\sigma = \sigma_b$ (there are no borders).

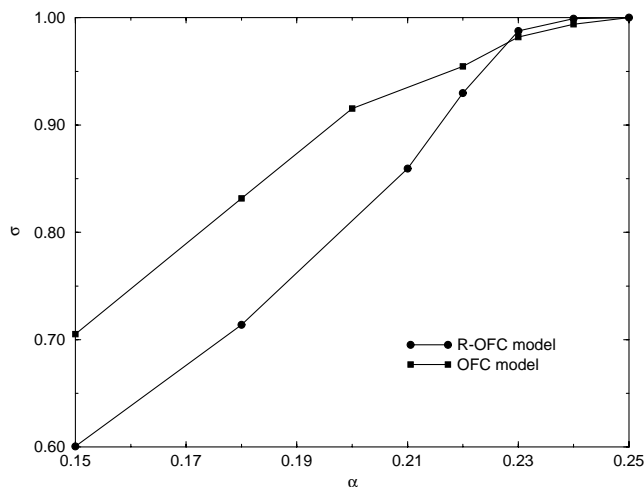


FIG. 1. Branching rate as a function of the conservation parameter α . Squares refer to the Olami-Feder-Christensen model (OFC) and the circles refer to the random version of the OFC (R-OFC) model. In all cases the lattice size is $L = 100$.

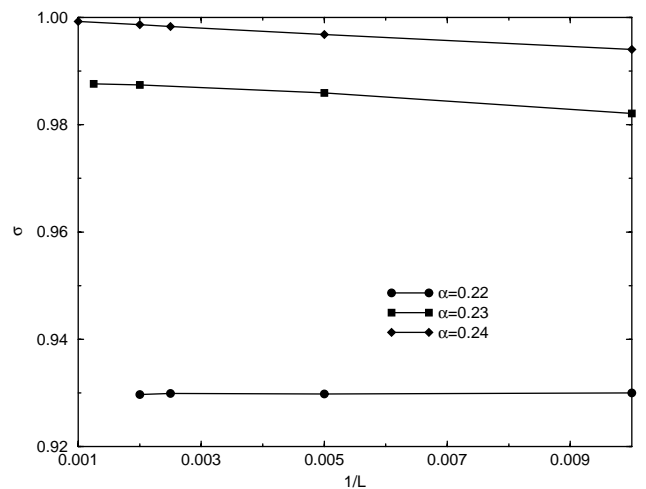


FIG. 2. Branching rate as a function of the inverse of lattice size ($1/L$) for the random version of the Olami-Feder-Christensen model. Different curves refer to different levels of conservation ($\alpha = 0.22, 0.23$, and 0.24). We can see that even for $\alpha = 0.24$, if we let $L \rightarrow \infty$, the branching rate σ tends to a value smaller than 1. The system shows a qualitatively different behavior only if $\alpha = 0.25$.

There are no relevant differences between the behaviors of the OFC and the R-OFC models. For both of them, $\sigma(\alpha) \rightarrow 1$ smoothly from below as $\alpha \rightarrow 0.25$, with no sign of any type of discontinuity in its behavior. Also, as can be seen in Fig. 1, $\sigma_{\text{OFC}} < \sigma_{\text{R-OFC}}$, for $0.23 \leq \alpha < 0.25$. From theoretical considerations [12], we know that $\sigma_{\text{R-OFC}} < 1$ for $\alpha < 0.25$.

In Figs. 2 and 3 we present the dependence of σ on the lattice size for the R-OFC and the OFC models. These figures show that σ grows almost linearly with $1/L$ with no suggestion that $\sigma \rightarrow 1$ as $1/L \rightarrow 0$. The behavior of the

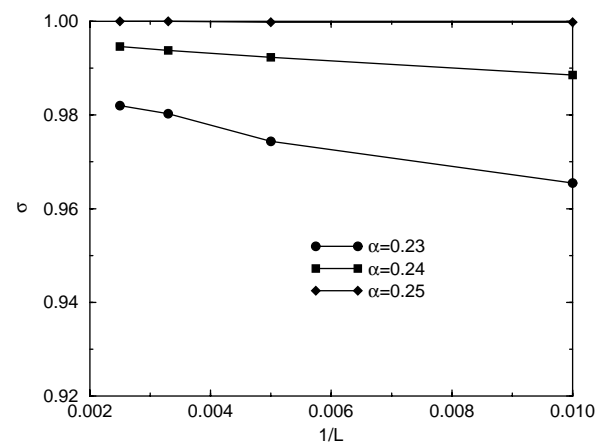


FIG. 3. The branching rate as a function of the inverse of lattice size ($1/L$) for the Olami-Feder-Christensen model. Different curves refer to different levels of conservation ($\alpha = 0.23, 0.24$, and 0.25). We can see that even for $\alpha = 0.24$, if we let $L \rightarrow \infty$, the branching rate σ tends to a value smaller than 1. Note that, in the conservative case ($\alpha = 0.25$), σ is almost 1.00 even to very small lattices.

system seems to be qualitatively different only if $\alpha = 0.25$ (conservative case).

We also checked the dependence of σ on the generation n within an avalanching process. We see that $\sigma(n)$ converges relatively fast to an asymptotic value [15]. None of our conclusions were affected if we considered these asymptotic values of $\sigma(n)$ instead of the average value.

The existence of SOC in the nonconservative regime of the OFC model has been accepted based mainly on numerical results of a Letter by Middleton and Tang [7] in 1995. In this paper, the authors showed how the natural tendency of this model to synchronize is destroyed by inhomogeneities introduced by the asymmetries of the boundaries, creating long-range correlations and leading to a power-law behavior in the distribution of avalanche sizes. The apparent contradiction between this result and ours can be understood from the conclusions of Kinouchi and Prado [8]. In this paper, the study of two different models with an analytical solution (the extremal Feder and Feder model (EFF) with and without noise), shows that the effect of noise is to enlarge the region where the system displays an apparent critical behavior, leading to what was called “almost criticality.” The EFF model with noise displays a power-law behavior (although it is not critical). In contrast, in the noiseless model, large avalanches occur in the conservative limit only. This also seems to be the case with the OFC model. The randomness introduced by the asymmetries of the boundaries creates correlations that enlarge the critical region, leading to an almost critical behavior, although it is not enough to ensure true criticality.

In conclusion, we showed that the analysis of $\sigma(\alpha)$ is a complementary approach to define if a model is or is not critical. This new method revealed that the behavior of the OFC model is qualitatively identical to the behavior of the R-OFC model. In contrast to previous results, the Olami-Feder-Christensen model seems to be critical only in the conservative regime, that is, for $\alpha = 0.25$. Both models are “almost” critical in the sense defined in Ref. [8]: $\sigma \approx 1$ when $\alpha \approx 0.25$, leading to a power-law behavior of the avalanche sizes for many decades, and making it (almost) impossible to distinguish this behavior from true self-organized criticality based on the observation of power laws and finite-size scaling fits.

Although, according to the present results, this seems highly unlikely, we point out that numerical arguments can never rule out the possibility of a true critical behavior in a very tiny region around the conservative limit. It should be mentioned although that previous numerical results suggest that this range is not so tiny at all, but rather large. Our results, on the other hand, are a clear indication that the nonconserving OFC model is not critical.

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*Email address: josue@if.usp.br

†Email address: prado@if.usp.br

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- [13] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967).
- [14] The R-OFC model follows the same dynamical rules of the OFC model, except for the way energy is distributed among the sites. When a site (i, j) becomes supercritical, instead of spreading its energy to the (four) nearest neighbors ($F_{nn} \rightarrow F_{nn} + \alpha F_{ij}$), it spreads energy to sites chosen at random. Each time a site relaxes, new “random neighbors” are chosen, in a process that has been called “annealed randomness” in literature. Such random versions have been constructed and studied because they can be thought of as numerical simulations of mean-field approaches of lattice models. For more details about the R-OFC model, see, for instance, the articles in Ref. [12].
- [15] The evolution of σ within an avalanching process is the current subject of another work and will be published elsewhere. We called each time step in an avalanching process a “generation.” Preliminary results indicate that $\sigma(n)$ converges relatively fast (ten generations are enough) to an asymptotic value slightly smaller than the average used in this paper, $\sigma = \langle n_d \rangle$. In the OFC model, for instance, we obtained, for $\alpha = 0.25$, $\sigma_\infty = 0.99787 \pm 0.00006$ and, for $\alpha = 0.24$, $\sigma_\infty = 0.9881 \pm 0.0006$.