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Weighted growing simplicial complexes

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Simplicial complexes describe collaboration networks, protein interaction networks, and brain networks and in general network structures in which the interactions can include more than two nodes. In real applications, often simplicial complexes are weighted. Here we propose a nonequilibrium model for weighted growing simplicial complexes. The proposed dynamics is able to generate weighted simplicial complexes with a rich interplay between weights and topology emerging not just at the level of nodes and links, but also at the level of faces of higher dimension.

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I. INTRODUCTION

Recently, generalized network structures such as multilayer networks [1,2] and simplicial complexes [3–5] have been attracting increasing attention in the network science community. In the past 20 years, very significant advances in the understanding of complex systems have been made using network theory [6–8]. Since the increasingly rich Big Data datasets on social, technological, and biological systems often include information that goes beyond what is possible to describe by a single network, it is now becoming of fundamental importance to characterize generalized network structures. For instance, in many cases data include a set of interactions having different connotations or occurring between more than two nodes. The flourishing field of multilayer networks [1,2] defines a way to treat networks where links have different connotations. Simplicial complexes [3–5], instead, allow for the description of interactions occurring between more than two nodes. As such, simplicial complexes have an important role to play when studying a large variety of complex systems. For example, in scientific collaboration networks, collaboration extends to teams of more than two scientists; in actor collaboration networks, it is rare for a cast of a movie to include only two actors. In on-line social networks, the rich structure of possible actions such as tagging, posting, or linking to other users also allows for the identification of interactions between more than two nodes. In biology, simplicial complexes are useful to understand protein interaction networks. In fact, in order to perform a function, proteins in the cell bind together to form protein complexes typically including more than two different types of interacting protein. Interestingly, one of the first algorithms proposed for detecting overlapping communities, namely the clique percolation algorithm [9,10], effectively uses cliques as simplicies to decompose the network in different mesoscopic clusters. Finally, simplicial complexes are becoming increasingly popular in analyzing brain networks [3], where one needs to distinguish, for instance, between three regions of the brain that are pairwise correlated, and the case in which they are typically activated all at the same time.

From a network perspective, simplicial complexes can be interchanged with hypergraphs [11,12] for the analysis of real networked datasets. However, simplicial complexes also have a geometric interpretation, and they can be interpreted as the result of gluing nodes, links, triangles, tetrahedra, etc. along their faces. As such, simplicial complexes can be used to characterize the resulting network geometry using, for instance, novel definitions of network curvatures [13–21] or characterizing their emergent geometrical properties [22,23]. Simplicial complexes are also starting to be widely used to perform topological analysis of network datasets and of dynamical processes defined on networks [3,24–26]. Most notably, this approach has been applied to brain functional networks showing that these novel techniques can reveal important differences between networked datasets that cannot be detected by more traditional methods [3,24,25].

Different frameworks have been proposed to model simplicial complexes. On one side, there are equilibrium models of static simplicial complexes that generalize the random graph or the configuration model to simplicial complexes [4,11,12,27–30]. On the other side, there are nonequilibrium models of growing simplicial complexes displaying emergent structural properties and geometry [22,23,31–33]. These models generalize at the same time growing network models [34,35] with preferential attachment and Apollonian networks [36–39].

In real applications, simplicial complexes are typically weighted, which explains the need to extend the modeling framework to characterize weighted simplicial complexes. For instance, in scientific collaboration networks, teams of collaborators can be weighted by the strength of their collaboration (i.e., how many papers a scientific collaboration has produced). Here we characterize a weighted simplicial network model using a nonequilibrium dynamics. The evolution of the topology of these networks is based on the recently proposed framework of Network Geometry with Flavor [31], which can be used to generate networks with different complex topology, including hyperbolic manifolds, scale-free networks, and networks with relevant modularity.

Here our focus will be on characterizing the rich interplay between weights and topology in these models. In single networks [40–43], it has been shown by analysis of a vast set of real datasets that weights might not be distributed uniformly over the links of the network. Specifically in some networks, hub nodes can have connections with on average stronger weights than the typical connections of low degree nodes. The way to characterize these weight-topology correlations is by studying the scaling of the average strength of nodes as a function of their degree. If the strength grows linearly with the degree, the weights are uniformly distributed among the nodes of the network. If instead the observed scaling is superlinear, then hubs typically have links with stronger weights than low degree nodes. Both linear and superlinear scaling have been observed in real-world networks [40]. While initially growing weighted network models appeared to capture only the linear scaling [41], it was later shown that the emergence of weight-topology correlations can be described in the framework of growing network models, including growth of the network by the continuous addition of new links, and at the same time an increase in the weights of the existing links driven by a reinforcement dynamics [42].

Here we study weight-topology correlations in growing simplicial complexes, showing that they emerge not just at the network level but also for δ-faces of higher dimension. Finally, we compare the results obtained in the mean-field approximation with extensive numerical simulations.

The paper is organized as follows. In Sec. II we define weighted simplicial complexes and their main structural properties. In Sec. III we define our model of growing weighted simplicial complexes. In Sec. IV we discuss the mean-field solution of the model. In Sec. V we compare our theoretical prediction with the results of the numerical simulations. Finally, in Sec. VI we present the conclusions. This paper is accompanied by a code published online [44].

II. SIMPLICIAL COMPLEXES

Let us consider N nodes $i = 1,2,\ldots,N$. A simplex of dimension d represents an interaction between a set of $d+1$ of these nodes. For instance, a simplex α of dimension d (also called d-simplex) is given by

$$\alpha = [i_0,i_1,\ldots,i_d], \tag{1}$$

where i_n with $n \in \{0,1,\ldots,d\}$ indicates a node of the simplex. A face α' of the d-simplex α is a δ-simplex with $0 \leqslant \delta < d$ formed by a subset of the nodes of α, i.e., $\alpha' \subset \alpha$. A simplex also has a geometrical interpretation and can be considered as a d-dimensional volume. This justifies the choice of calling its subsets its "faces." For example, a simplex of dimension $d = 2$ is a triangle, and all its links and nodes form its faces. Similarly, a simplex of dimension $d = 3$ is a tetrahedron and its faces include four triangles, six links, and six nodes.

A simplicial complex $\mathcal{K}$ of dimension d is a collection of simplices of at most dimension d glued along their shared faces. Additionally, every simplicial complex $\mathcal{K}$ satisfies the following constraint: if a simplex belongs to the simplicial complex (i.e., $\alpha \in \mathcal{K}$), then the simplicial complex also includes all of the faces $\alpha' \subset \alpha$ of that simplex (i.e., $\alpha' \in \mathcal{K}$). In order words, the simplicial complex is closed under the operation of inclusion of faces of its simplices.

In this paper, we indicate with $Q_{d,\delta}(N)$ the set of all possible δ-dimensional faces (δ-faces) in a d-dimensional simplicial complex formed by N nodes. Additionally, we indicate with $S_{d,\delta}$ the set of δ-faces belonging to the simplicial complex under consideration. Here we consider exclusively d-dimensional simplicial complexes constructed by gluing d-dimensional simplices. The structure of such d-dimensional simplicial complexes of N nodes is determined by the adjacency tensor $\mathbf{a}$ with elements $a_\alpha = 1,0$ indicating whether the simplex $\alpha \in Q_{d,d}(N)$ is present ($a_\alpha = 1$) or absent ($a_\alpha = 0$) from the simplicial complex, i.e.,

$$a_\alpha = \begin{cases} 1 & \text{if} \quad \alpha \in S_{d,d}, \\ 0 & \quad\ \ \text{otherwise.} \end{cases}$$

The weights of the simplices are indicated by the weight tensor $\mathbf{w}$, with elements w_α indicating the weight of simplex α. In a simplicial complex representing co-authorship, for example, a simplex represents a set of co-authors that have collaborated on at least one paper together, while the weight of that simplex corresponds to the total number of papers that have been co-authored by the team. To characterize the properties of the simplicial complex, we use here the *generalized degrees* and *generalized strengths* of the δ-faces.

The generalized degree $k^{\alpha}_{d,\delta}$ of a δ-face $\alpha \in S_{d,\delta}$ [30,31,33] is the number of d-dimensional simplices incident to it,

$$k^{\alpha}_{d,\delta} = \sum_{\alpha' \in Q_{d,d}(N) | \alpha' \supseteq \alpha} a_{\alpha'}. \tag{2}$$

The generalized strength $s_{d,\delta}(\alpha)(t)$ of a δ-face $\alpha \in S_{d,\delta}$ is the sum of the weights of the d-dimensional simplices incident to it,

$$s^{\alpha}_{d,\delta} = \sum_{\alpha' \in Q_{d,d}(N) | \alpha' \supseteq \alpha} a_{\alpha'} w_{\alpha'}. \tag{3}$$

The generalized degree of a δ-face α is related to the generalized degree of the δ'-dimensional faces incident to it, with $\delta' > \delta$, by the simple combinatorial relation [30]

$$k^{\alpha}_{d,\delta} = \frac{1}{\binom{d-\delta}{\delta'-\delta}} \sum_{\alpha' \in S_{d,\delta'} | \alpha' \supseteq \alpha} k^{\alpha'}_{d,\delta'}. \tag{4}$$

Moreover, since every d-dimensional simplex belongs to $\binom{d+1}{\delta+1}$ δ-dimensional faces, in a simplicial complex with M d-dimensional simplices we have

$$\sum_{\alpha \in S_{d,\delta}} k^{\alpha}_{d,\delta} = \binom{d+1}{\delta+1} M. \tag{5}$$

Interestingly, the generalized strength a δ-face α satisfies also

$$s^{\alpha}_{d,\delta} = \frac{1}{\binom{d-\delta}{\delta'-\delta}} \sum_{\alpha' \in S_{d,\delta'} | \alpha' \supseteq \alpha} s^{\alpha'}_{d,\delta'}. \tag{6}$$

The skeleton of a simplicial complex is the network formed by all its 0-faces (nodes) and 1-faces (links). To a weighted simplicial complex we can associate a skeleton that is a weighted network in which the weights ω_{ij} of the links in the skeleton are equal to the generalized strengths of the links in the simplicial complex, i.e.,

$$\omega_{ij} = s^{[i,j]}_{d,1}, \tag{7}$$

and the strength S_i of a node i is

$$S_i = \sum_{j=1}^{N} \omega_{ij}. \tag{8}$$

The generalized degrees strength $s^{[i]}_{d,0}$ of the node i of the simplicial complex is naturally related with the strength S_i of the node in the skeleton network. Using Eqs. (6) and (7), it is possible to see that

$$S_i = d s_{d,0}^{[i]}. \tag{9}$$

Instead, in general the only relation between the generalized degree $k_{d,0}^{[i]}$ of node I and the degree K_i of the same node in the skeleton network is

$$K_i \leqslant d k_{d,0}^{[i]} \tag{10}$$

because some of the d-simplices incident to the node i might share some links (1-faces).

In weighted networks, it has been shown that it is possible to characterize the interplay between the network topology and the weights of the links by classifying networks depending on the scaling of the strength as a function of the degree of the nodes. Specifically, it has been shown that for some networks the weights of the links are distributed rather uniformly, resulting in a linear dependence of the strength of the nodes with its degree,

$$S_i \propto K_i, \tag{11}$$

while in other networks hub nodes tend to have links with higher weights than low degree nodes. This latter scenario results in a superlinear scaling of the strength versus the degree, i.e.,

$$S_i \propto (K_i)^\theta, \tag{12}$$

with $\theta > 1$. An example of networks with linear dependence of the strength versus degree are collaboration networks, while an example of a nonlinear dependence of strength on degree are airport networks where the weights measure the number of passengers for each flight connection. In Ref. [42] it has been shown that a simple growing network model with reinforcement of the links is actually able to generate networks with linear and superlinear scaling of the strength versus degree, depending on the rate at which new links are added with respect to the rate at which links are reinforced.

Here we are proposing a model for growing simplicial complexes showing a very rich phenomenology, and we show evidence that in simplicial complexes it is possible to characterize the correlations between weights and topology by exploring the dependence of the generalized strength $s_{d,\delta}^\alpha$ versus the generalized degree $k_{d,\delta}^\alpha$. Specifically, we are able to predict three alternative possible scalings: linear, superlinear, and exponential, i.e.,

$$s_{d,\delta}^\alpha \propto \begin{cases} k_{d,\delta}^\alpha, \\ \left(k_{d,\delta}^\alpha\right)^\theta, \\ \exp\left[\beta k_{d,\delta}^\alpha\right], \end{cases} \tag{13}$$

with $\theta > 1$ and β indicating a constant greater than zero. In this case, the superlinear scaling indicates weight-topology correlations, and these correlations are even more pronounced for the exponential scaling.

III. THE MODEL

In this section, we present a model of growing weighted simplicial complexes based on reinforcement of the weights of the simplices and capable of displaying important weight-topology correlations depending on the value of its parameters. This model is based on the already proposed model of Network Geometry with Flavor [31], but it includes two important new elements with respect to the mentioned model: (i) the simplicial complexes generated by this model are weighted, and (ii) the simplicial complexes generated by this model can have nontrivial homology.

In this model, the weighted growing simplicial complexes are generated as follows. We start at time $t = 1$ from an initial finite simplicial complex that comprises $m_0 > m$ d-dimensional simplices of total weight ω_0. At each time step $t > 1$, two processes take place:

(A) *Add m simplices*: A new node arrives and m new d-simplices with initial weight w_0 are created between the node and preexisting $(d-1)$ faces. The probability $\Pi_{d-1}(\alpha)$ that a given $(d-1)$ face α is selected is given by

$$\Pi_{d-1}(\alpha) = \frac{1}{\mathcal{Z}_t}(1 + s n_\alpha), \tag{14}$$

where $n_\alpha = k_{d,d-1}^\alpha - 1$ is called the occupancy number and where s is a parameter called *flavor*, which takes the values $s = -1,0,1$ and controls the simplicial complex topology. Note that in Eq. (14), $\mathcal{Z}_t$ is a normalization constant given by $\mathcal{Z}_t = \sum_{\mu \in S_{d,d-1}(t)}(1 + s n_\alpha)$.

(B) *Reinforce m' simplices*: At this step, m' existing d-simplices are selected and their weights are increased by w_0. A d-simplex α with weight w_α is selected for reinforcement with probability $\tilde{\Pi}_d(\alpha)$ proportional to its weight, i.e.,

$$\tilde{\Pi}_d(\alpha) = \frac{w_\alpha}{\tilde{\mathcal{Z}}_t}, \tag{15}$$

where $\tilde{\mathcal{Z}}_t = \sum_{\alpha \in S_{d,d}(t)} w_\alpha$.

In Fig. 1, we describe the two processes (processes A and B) for a two-dimensional simplicial complex starting from a given initial condition. The flavor s has an important effect on the topological properties of the simplicial complexes produced. Selection of $s = -1$ imposes the constraint that the generalized degree $k_{d,d-1}(\alpha)$ of a $(d-1)$ face α can only take the values 1 and 2, or equivalently it imposes that n_α can only take values 0 and 1, which leads to the simplicial complex produced being a d-dimensional manifold. Choosing $s = 0$ or 1 removes this constraint, and it gives a selection probability $\Pi_{d-1}(\alpha)$ that is uniform on the set of all $(d-1)$ faces for $s = 0$ and a form of preferential attachment with $\Pi_{d-1}(\alpha) \propto k_{d,d-1}^\alpha$ for $s = 1$.

In Fig. 2, we plot the weighted skeleton networks of two simplicial complexes generated by the model in the case $d = 3$ and $s = -1$ for $(m,m') = (1,2)$ and $(m,m') = (2,1)$. The weights of the links in these networks indicate the generalized strengths of the links in their corresponding simplicial complexes. While in the case $(m,m') = (1,2)$ nodes with high degree have typically links with stronger weights than the weights of low-degree nodes, the weights are more homogeneously distributed in the case $(m,m') = (2,1)$.

We note that for the case $m = 1$, each addition of a new node and its initial d-dimensional simplex does not change the Euler characteristics χ, i.e., indicating with $\chi(t)$ the Euler characteristics at time t, we get

$$\Delta\chi(t) = \chi(t) - \chi(t-1) = 0. \tag{16}$$

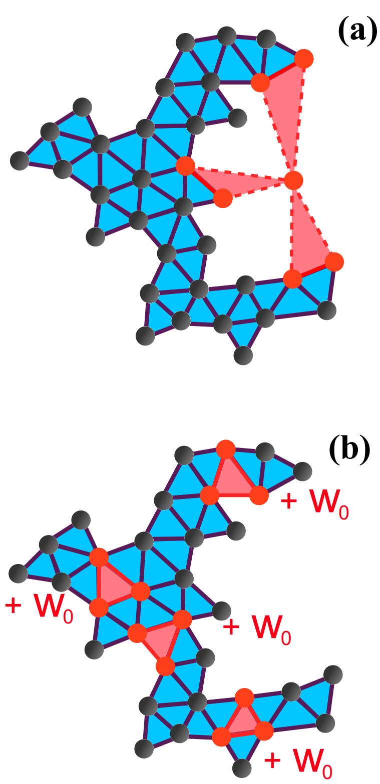

FIG. 1. Graphical representation of process A [panel (a)] and process B [panel (b)] for a two-dimensional simplicial complex with $m = 3$ and $m' = 4$ starting from a given initial condition.

Therefore, if the initial condition has $\chi(0) = 1$, we obtain $\chi(t) = 1$ for every time t and we have a trivial topology. However, for the case $m > 1$, the Euler characteristics change with time according to

$$\Delta\chi(t) = 1 - m < 0, \tag{17}$$

and therefore it could be very interesting to study in more detail the topology of these networks. Additionally, the information on the weights could be exploited by considering, as in Refs. [24,25], the persistent homology induced by a weights-based filtration of the simplicial complex.

IV. MEAN-FIELD SOLUTION OF THE MODEL

A. Mean-field solutions for the generalized degrees

In this section, our aim is to derive the time evolution of the generalized degrees of the δ faces of the simplicial complexes using a mean-field approximation. Toward that end, let us define the probability $\Pi_\delta(\alpha)$ that due to the addition of a single new simplex in the simplicial complex (process A), the δ-face α increases its generalized degree. The probability $\Pi_\delta(\alpha)$ is the sum of the probabilities that any $(d-1)$ face $\alpha' \supseteq \alpha$ is chosen for attaching a new simplex, i.e.,

$$\Pi_\delta(\alpha) = \sum_{\alpha' \in S_{d,d-1} | \alpha' \supseteq \alpha} \Pi_{d-1}(\alpha') = \frac{1}{\mathcal{Z}_t} \sum_{\alpha' \in S_{d,d-1} | \alpha' \supseteq \alpha} 1 - s + s k^{\alpha'}_{d,d-1}. \tag{18}$$

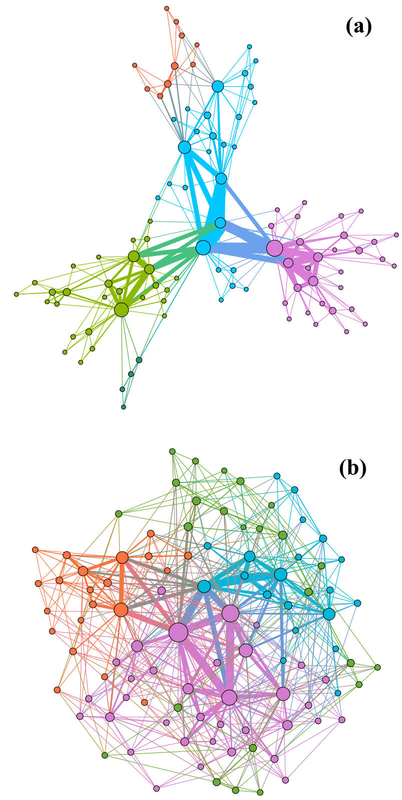

FIG. 2. Skeleton networks of simplicial complexes generated by the model for $d = 3$, $N = 100$, and $s = -1$. Node sizes indicate their degrees while link widths indicate their generalized strength. Node and edge colorings indicate community structure calculated according to the Louvain algorithm [45]. Panel (a) shows the skeleton of a simplicial complex with $m = 1$ and $m' = 2$, while panel (b) shows the skeleton of a simplicial complex with $m = 2$ and $m' = 1$.

To derive this expression explicitly in terms of the generalized degree of α, we use the following relation:

$$\sum_{\alpha' \in S_{d,d-1} | \alpha' \supseteq \alpha} 1 - s + s k^{\alpha'}_{d,d-1} = (1-s)c_\delta + (d + s - \delta - 1) k^{\alpha}_{d,\delta}, \tag{19}$$

where

$$c_\delta = \begin{cases} 1 & \text{for} \quad \delta > 0, \\ m & \text{for} \quad \delta = 0. \end{cases} \tag{20}$$

To see why Eq. (19) holds, observe that Eq. (4) implies the following:

$$\sum_{\alpha' \in S_{d,d-1} | \alpha' \supset \alpha} k^{\alpha'}_{d,d-1} = (d - \delta) k^{\alpha}_{d,\delta}. \tag{21}$$

Additionally, the following relation can also be shown to hold for our model:

$$\sum_{\alpha' \in S_{d,d-1} | \alpha' \supset \alpha} 1 = \begin{cases} 1 + (d - \delta - 1) k^{\alpha}_{d,\delta} & \text{for} \quad \delta > 0, \\ m + (d-1) k^{\alpha}_{d,\delta} & \text{for} \quad \delta = 0. \end{cases} \tag{22}$$

This equation can be derived by observing that each initial d-simplex, which includes the δ-face α, contributes by $d - \delta$ to the sum on the left-hand side of Eq. (22). In fact, there are $\binom{d-\delta}{d-\delta-1} = d-\delta$ ways to choose the $d-(1+\delta)$ nodes of a $(d-1)$ face that do not belong to the δ-face out of the $d-\delta$ nodes of the d-simplex that do not belong to the δ face. Initially, any δ-face with $\delta>0$ belongs to a single simplex, while the δ-faces with $\delta=0$ (the nodes) belong to m simplices. Finally, every d-simplex that further increases the generalized degree of the δ-face contributes to the sum just by $d-\delta-1$ because simplices are glued along $(d-1)$ faces of the simplicial complex.

By using Eq. (19), we can express $\Pi_\delta(\alpha)$ in terms of the generalized degree $k^{\alpha}_{d,\delta}$ as

$$\Pi_\delta(\alpha) = \frac{(1-s)c_\delta + (d+s-\delta-1)k^{\alpha}_{d,\delta}}{\mathcal{Z}_t}. \tag{23}$$

From this expression, the mean-field equations for the generalized degree can be easily derived. In the mean-field approximation, the generalized degrees of the δ-faces are approximated with their average value over different stochastic realizations of the simplicial complex. Additionally, using a very well-established framework for simple networks [6–8], we will consider a continuous-time approximation in which the (average) degree $k_{d,\delta}(t,t_\alpha)$ that a δ-face α arrived in the network at time t_α has at time t is determined by deterministic differential equations. These equations read for any $0 \leqslant \delta \leqslant d-1$

$$\frac{\partial}{\partial t}k_{d,\delta}(t,t_\alpha) = m\Pi_\delta(\alpha), \tag{24}$$

where $\Pi_\delta(\alpha)$ is given by Eq. (23). Let us now note that the normalization constant $\mathcal{Z}_t$ is simply given, in the limit $t \gg 1$, by

$$\mathcal{Z}_t = \sum_{\alpha' \in S_{d,d-1}} 1 - s + s k^{\alpha'}_{d,d-1} \simeq m(d+s)t. \tag{25}$$

In fact, the total number of $(d-1)$ faces $\sum_{\alpha'\in S_{d,d-1}} 1 \simeq mdt$ for $t \gg 1$ because at each time we add m new d-dimensional simplices, each one contributing d new $(d-1)$ faces. Additionally, we have that $\sum_{\alpha'\in S_{d,d-1}} k^{\alpha'}_{d,d-1} \simeq m(d+1)t$ for $t \gg 1$ because any new simplex increases by 1 the generalized degree of each of its $(d+1)(d-1)$ faces. Therefore, using Eqs. (25) and (23) we can derive that, for sufficiently large times, the mean-field equation determining the generalized degree dynamics is given by

$$\frac{\partial k_{d,\delta}(t,t_\alpha)}{dt} = \frac{(1-s)c_\delta + (d+s-\delta-1)k_{d,\delta}(t,t_\alpha)}{(d+s)t}, \tag{26}$$

with the initial condition

$$k_{d,\delta}(t_\alpha,t_\alpha) = c_\delta = \begin{cases} 1 & \text{for} \quad \delta>0, \\ m & \text{for} \quad \delta = 0. \end{cases} \tag{27}$$

The solution of this equation is

$$k_{d,\delta}(t,t_\alpha) = \begin{cases} c_\delta \frac{d-\delta}{d+s-\delta-1}\left(\frac{t}{t_\alpha}\right)^{\lambda_\delta} + c_\delta\frac{s-1}{d+s-\delta-1} & \text{for} \quad \delta - s \neq d-1, \\ c_\delta \frac{1-s}{d+s}\log\left(\frac{t}{t_\alpha}\right) + c_\delta & \text{for} \quad \delta - s = d-1, \end{cases} \tag{28}$$

with

$$\lambda_\delta = \frac{d+s-\delta-1}{d+s}. \tag{29}$$

TABLE I. Distribution of generalized degrees of faces of dimension δ in a d-dimensional NGF of flavor s at $\beta=0$. For $d \geqslant d_c^{[\delta,s]} = 2(\delta+1)-s$, the power-law distributions are scale-free, i.e., the second moment of the distribution diverges.

Flavor	$s=-1$	$s=0$	$s=1$
$\delta = d-1$	Bimodal	Exponential	Power law
$\delta = d-2$	Exponential	Power law	Power law
$\delta \leqslant d-3$	Power law	Power law	Power law

The generalized degree distribution $P_{d,\delta}(k)$ for $\delta=d-1$ and $s=-1$ is bimodal, because only the generalized degrees $k^{\alpha}_{d,d-1}=1,2$ are allowed. For all the other cases, it is possible to derive the generalized degree distribution using the mean-field solution given by Eq. (28). In this way, it is found that the generalized degree distribution is exponential for $d-1+s-\delta=0$ and power-law for $d-1+s-\delta>0$. To derive these results, let us note that since at each time we add a constant number of δ-faces, the explicit expression for the probability $\hat{P}_\delta(t_\alpha<\tau)$ that a random δ-face has been added at time $t_\alpha<\tau$ is given by

$$\hat{P}_\delta(t_\alpha < \tau) = \frac{\tau}{t}. \tag{30}$$

Using this result and Eq. (28), the probability that the generalized degree $k_{d,\delta}(t,t_\alpha)$ is greater than k can be calculated to be given by

$$P(k_{d,\delta}(t,t_\alpha) \geqslant k) = \begin{cases} \left(\frac{c_\delta(d-\delta)}{k(d+s-\delta-1)}\right)^{\frac{1}{\lambda_\delta}} & \text{for} \quad \delta - s < d-1, \\ \exp\left[-\frac{d+s}{(1-s)c_\delta}k\right] & \text{for} \quad \delta - s = d-1. \end{cases} \tag{31}$$

This leads to the following generalized degree distribution $P_{d,\delta}(k)$:

$$P_{d,\delta}(k) = -\frac{dP(k_{d,\delta}(t,t_\alpha)\geqslant k)}{dk} = \begin{cases} \frac{d+s}{d+s-\delta-1}\left(c_\delta\frac{d-\delta}{d+s-\delta-1}\right)^{\frac{1}{\lambda_\delta}} k^{-\frac{1}{\lambda_\delta}-1} & \text{for} \quad \delta - s < d-1, \\ \frac{d+s}{(1-s)c_\delta}\exp\left[-\frac{d+s}{(1-s)c_\delta}k\right] & \text{for} \quad \delta - s = d-1, \end{cases} \tag{32}$$

valid as long as $\delta - s < d$. Therefore, the generalized degree distribution of δ-dimensional simplices in growing simplicial networks with flavor s follows Table I. Additionally, the generalized degree distribution $P_{d,\delta}(k)$ given by Eq. (32) decays as a power-law $P_{d,\delta}(k) \propto k^{-\gamma_{d,\delta}}$ with a power-law exponent

$$\gamma_{d,\delta} = 1 + \frac{1}{\lambda_\delta} = 1 + \frac{d+s}{d+s-\delta-1} \tag{33}$$

as long as $\delta - s < d-1$. These distributions are scale-free if $\gamma_{d,\delta} \leqslant 3$, or equivalently they are scale-free if

$$d \geqslant d_c^{[\delta,s]} = 2(\delta+1)-s. \tag{34}$$

Let us now observe in the considered growing simplicial complex the degree of a node K_i not belonging to the set of nodes in the initial condition, which is given by

$$K_i = k^{i}_{d,0} + (d-1)m. \tag{35}$$

In fact, initially each node has degree dm, and subsequently the degree increases by 1 for any d-simplex glued to one of the $(d-1)$ faces of the node. It follows then that if the generalized degree distribution of the nodes is scale-free, the degree distribution of the skeleton network is also scale-free. As a result, growing simplicial complexes of flavor $s=1$ are scale-free for any $d \geqslant 1$, those of flavor $s=0$ are scale-free for $d \geqslant 2$, and those of flavor $s=-1$ are scale-free for $d \geqslant 3$.

B. Probability of a simplex

In this section, we derive the probability of a δ-simplex in terms of the arrival times of its nodes. Let us indicate each δ-face α_δ as the sequence of its nodes $\alpha_\delta = [j_0, j_1, \ldots, j_\delta]$ where the nodes are ordered according to the time of their arrival in the simplicial complex, i.e., $t_{j_0} < t_{j_1} < \cdots < t_{j_\delta}$. A new node appears in the simplicial complex at every time step and forms d-dimensional simplices with m already existing $(d-1)$ faces. Thus, the δ-face α_δ is the result of the subsequent addition of new d-dimensional simplices to the faces $\alpha_{\delta'} = [j_0, j_1, \ldots, j_{\delta'}]$ formed by the $\delta'+1$ oldest nodes of α_δ for each $0 \leqslant \delta' < \delta$. Specifically, after node j_0 is added to the network at time t_{j_0}, we must have that the node j_1, which arrived at the simplicial complex at time t_{j_1}, belongs to a new d-simplex incident to the node j_0. Subsequently node j_2, which arrived in the simplicial complex at time t_{j_2}, should belong to a d-simplex incident to the face $\{j_0, j_1\}$, and so on. Therefore, probability p_{α_δ} that the δ-face α_δ belongs to the simplicial complex may be written

$$p_{\alpha_\delta} = \prod_{n=0}^{\delta-1} \pi_n(t_{j_{n+1}}, t_{j_n}), \tag{36}$$

where $\pi_n(t_{j_{n+1}}, t_{j_n})$ is the probability that the j_{n+1} node that arrived in the simplicial complex at time $t_{j_{n+1}}$ belongs to a d-simplex incident to the face α_n formed by the set of nodes $\{j_0, j_1, \ldots, j_n\}$ of arrival times $t_{j_0} < t_{j_1} < \cdots < t_{j_n}$.

Let us observe that in the mean-field approximation, as we have shown in the previous section, the generalized degree $k_{d,\delta}^{\alpha_\delta}$ of the δ-face α_δ only depends on the time t_{j_δ} of arrival of the younger node of the simplex, i.e., $k_{d,\delta}^{\alpha_\delta} = k_{d,\delta}(t, t_{j_\delta})$. Therefore, $\pi_\delta(t_{j_{\delta+1}}, t_{j_\delta})$ is given by

$$\pi_\delta(t_{j_{\delta+1}}, t_{j_\delta}) = \frac{(1-s)c_\delta + (d+s-\delta-1)k_{d,\delta}(t_{j_{\delta+1}}, t_{j_\delta})}{(d+s)t_{j_{\delta+1}}},$$

where we have used $\pi_\delta(t_{j_{\delta+1}}, t_{j_\delta}) = m\Pi_\delta(\alpha_\delta)$ and the expression of Π_δ given by Eqs. (23) and (25). Replacing $k_{d,\delta}^{\alpha_\delta}(t_{j_{\delta+1}})$ with the mean-field (expected) generalized degree $k_{d,\delta}(t_{j_{\delta+1}}, t_{j_\delta})$ given by Eq. (28), we obtain

$$\pi_\delta(t_{j_{\delta+1}}, t_{j_\delta}) = c_\delta \frac{d-\delta}{d+s} t_{j_\delta}^{\frac{1+\delta}{d+s}-1} t_{j_{\delta+1}}^{-\frac{1+\delta}{d+s}}. \tag{37}$$

Finally, using Eqs. (37) and (36) we get a closed expression for the probability p_{α_δ} of a δ-face as a function of the times $\{t_{j_1}, t_{j_2}, \ldots, t_{j_\delta}\}$ of arrival of its nodes in the simplicial complex, given by

$$p_{\alpha_\delta} = m \frac{d!}{(d-\delta)!(d+s)^\delta} (t_{j_0} t_{j_1}, \ldots, t_{j_{\delta-1}})^{\frac{1}{d+s}-1} t_{j_\delta}^{-\frac{\delta}{d+s}}. \tag{38}$$

C. Mean-field solution for the weight of a simplex

Here we derive a mean-field expression for the (average) weight $w(t, t_\alpha)$ that the d-dimensional simplex α added to the simplicial complex at time t_α has at time t.

Since according to process B at each time we reinforce m' random simplices increasing their weight by w_0, we have

$$\frac{\partial w(t, t_\alpha)}{\partial t} = w_0 m' \tilde{\Pi}_d(\alpha), \tag{39}$$

where $\tilde{\Pi}_d(\alpha) = w_\alpha(t)/\tilde{\mathcal{Z}}_t$ is the probability that the d-simplex α is reinforced at time t. This equation has the initial condition

$$w(t_\alpha, t_\alpha) = w_0 \tag{40}$$

since each new simplex initially has weight w_0. At each time, m new simplices, each of weight w_0, are added to the simplicial complex, and m' existing simplices increase their weight by w_0. Therefore, we have that the normalization constant $\tilde{\mathcal{Z}}_t$ is given by

$$\tilde{\mathcal{Z}}_t = \sum_{\alpha \in S_{d,d}(t)} w_\alpha(t) = (m'+m)w_0 t + \omega_0 \simeq (m'+m)w_0 t,$$

where the last expression is valid for $t \gg 1$. It results that the mean-field Eq. (39) for the weights can also be written as

$$\frac{\partial w(t, t_\alpha)}{\partial t} = \lambda \frac{w(t, t_\alpha)}{t}, \tag{41}$$

where

$$\lambda = \frac{m'}{m+m'}. \tag{42}$$

Given the initial condition expressed by Eq. (40), this equation has the solution

$$w(t, t_\alpha) = w_0 \left(\frac{t}{t_\alpha}\right)^\lambda. \tag{43}$$

D. Mean-field approach for the generalized strengths

In this section, we evaluate the generalized strength of a δ-face in the mean-field approximation. In the spirit of the mean-field approximation, i.e., neglecting fluctuations, we identify the generalized strength $s_{d,\delta}^\alpha$ with its expected value $s_{d,\delta}(t, t_\alpha)$ over different simplicial complex realizations and conditioned on the existence of the face α with arrival time t_α. This is given by

$$s_{d,\delta}(t, t_\alpha) = \frac{\sum_{\alpha' \in Q_{d,d}(N) | \alpha' \supset \alpha} p_{\alpha'} w(t, t_{\alpha'})}{p_\alpha}. \tag{44}$$

Let us indicate each δ-simplex α by the set of its nodes $[i_0, i_1, \ldots, i_\delta]$ ordered according to the arrival times in the simplicial complex $t_{i_0} < t_{i_1} < \cdots < t_{i_\delta} = t_\alpha$. Similarly, we will indicate each d-simplex α' by the ordered set of its nodes $[j_0, j_1, \ldots, j_d]$ ordered according to the arrival times in the simplicial complex $t_{j_0} < t_{j_1} < \cdots < t_{j_d} = t_{\alpha'}$. When the δ-simplex α is a face of the d-simplex α', we have $[i_0, i_1, \ldots, i_\delta] \subset [j_0, j_1, \ldots, j_n]$. In this case, we may indicate with $q(r)$ the index of the node $i_r \subset \alpha$ in the list $[j_0, j_1, \ldots, j_n]$ specifying the nodes of the face α'. Therefore, we have

$$i_r = j_{q(r)}. \tag{45}$$

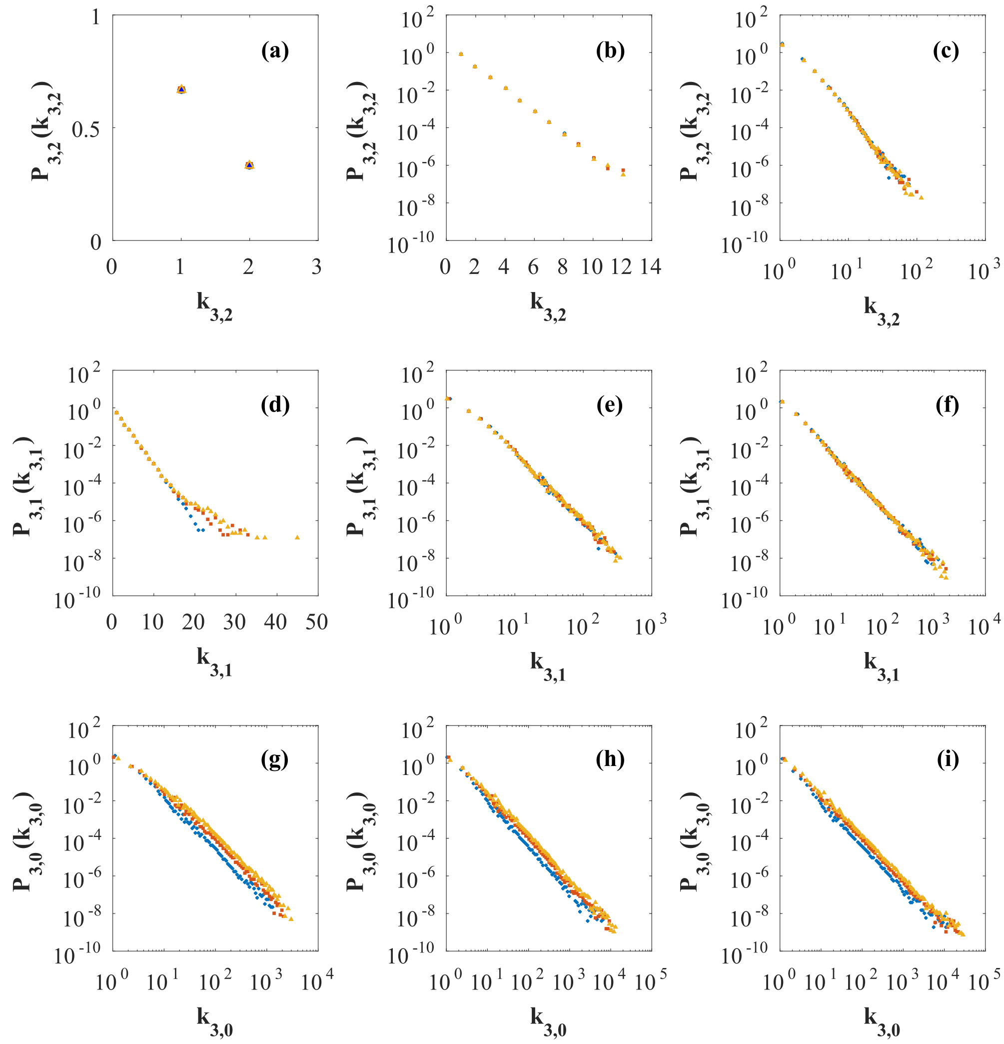

FIG. 3. Generalized degree distributions $P_{d,\delta}(k_{d,\delta})$ are shown for simplicial complexes of dimension $d=3$ and flavor $s=-1$ [panels (a), (d), and (g)], $s=0$ [panels (b), (e), and (h)], and $s=1$ [panels (c), (f), and (i)], and for faces of dimension $\delta=2$ [panels (a), (b), and (c)], $\delta=1$ [panels (d), (e), and (f)], and $\delta=0$ [panels (g), (h), and (i)]. The results of simulations are shown for $m=1$, 2, and 3 (blue circles, red squares, and yellow triangles, respectively). The simulated simplicial complexes have $N=10^5$ nodes, and the results are averaged over 10 simplicial complex realizations.

To be concrete, let us consider an example. In a simplicial complex of dimension $d=4$, consider the 4-simplex α',

$$\alpha'=[j_0,j_1,j_2,j_3,j_4]=[5,7,11,19,25] \tag{46}$$

and the 1-face α,

$$\alpha=[i_0,i_1]=[7,19]. \tag{47}$$

Since $i_0=j_1$ and $i_1=j_3$, we have

$$q(0)=1,\ q(1)=3. \tag{48}$$

In Eq. (44), let us now distinguish between contributions to the average generalized strength $s_{d,\delta}(t,t_\alpha)$ from d-simplices that contain the nodes of α in the positions specified by distinct $\{q(r)\}_{r=0,1,\ldots,\delta}$. Additionally, noting that $p_{\alpha'}$ in the mean-field approximation depends only on the set of arrival times of its nodes, and that each node is uniquely identified by its arrival time, and also taking the continuous approximation for arrival times t_{j_n}, we get the following expression for the average generalized strengths:

$$s_{d,\delta}(t,t_\alpha)=\frac{1}{p_{[i_0,\ldots,i_\delta]}}\sum_{\{q(r)\}_{r=0,1,\ldots,\delta}}\int_{t_{j_0}<\cdots<t_{j_d}}\left[\prod_{n=0}^{d}dt_{j_n}\right]\times\prod_{r=0}^{\delta}\hat{\delta}(t_{j_{q(r)}},t_{i_r})\,p_{[j_0,\ldots,j_d]}\,w(t,t_{j_d}), \tag{49}$$

where $\hat{\delta}(x,y)$ indicates the Kronecker delta. Using Eq. (38) for the probability $p_{[j_0,\ldots,j_d]}$ and Eq. (43) for the analytical expression of $w(t,t_d)$, we get

$$s_{d,\delta}(t,t_\alpha)=\frac{1}{p_\alpha}w_0m\frac{d!}{(d+s)^d}t^{\lambda}(t_{i_0}t_{i_1},\ldots,t_{i_\delta})^{\frac{1}{d+s}-1}\times\sum_{\{q\}}A_{q(\delta)}\left(\prod_{r=0}^{\delta-1}X_{q(r),q(r+1)}\right)B_{q(0)}, \tag{50}$$

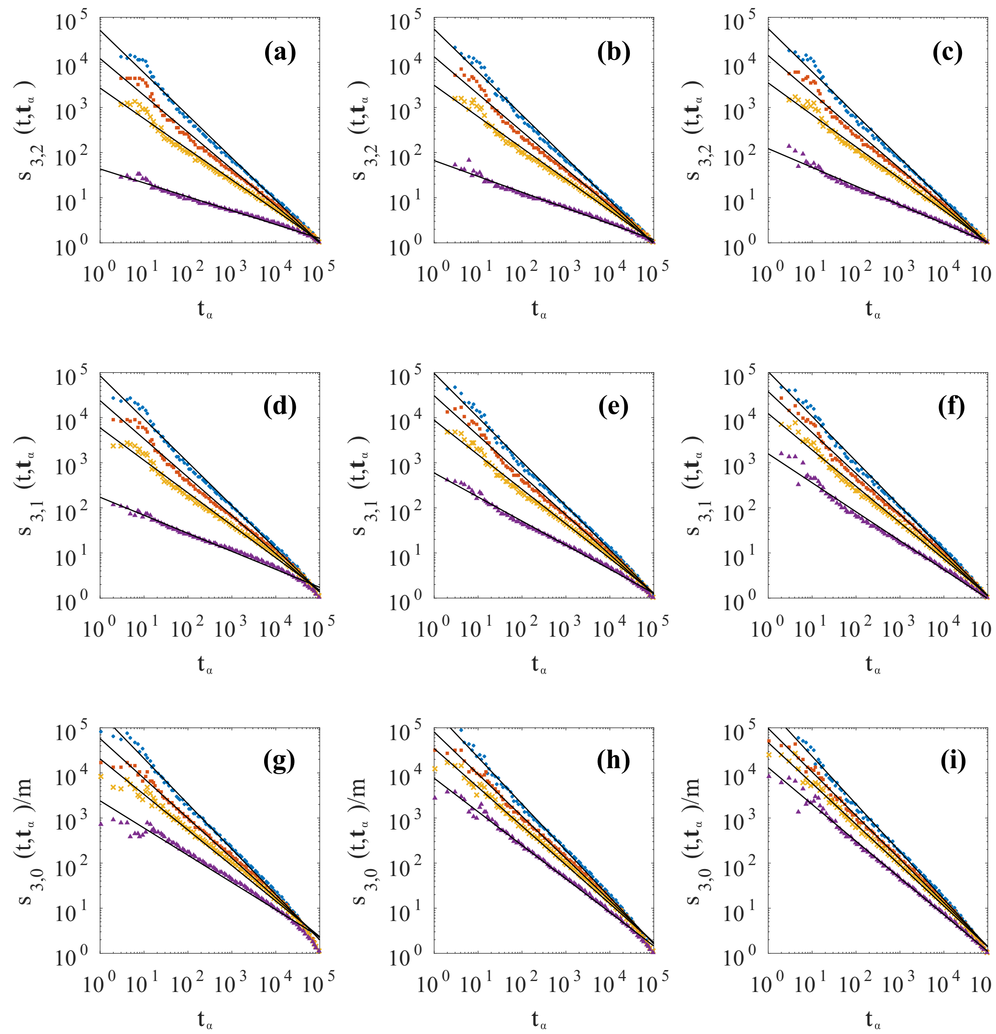

FIG. 4. The average generalized strengths $s_{d,\delta}(t,t_\alpha)$ of the δ-faces arrived in the network at time t_α are shown for simplicial complexes of dimension $d=3$, and flavor $s=-1$ [panels (a), (d), and (g)], $s=0$ [panels (b), (e), and (h)], and $s=1$ [panels (c), (f), and (i)] and for faces of dimension $\delta=2$ [panels (a), (b), and (c)], $\delta=1$ [panels (d), (e), and (f)], and $\delta=0$ [panels (g), (h), and (i)]. The results of simulations are shown for $(m=1,m'=5)$, $(m=2,m'=5)$, $(m=2,m'=3)$, and $(m=3,m'=1)$ (blue circles, red squares, yellow crosses, and purple triangles, respectively). The simulated simplicial complexes have $N=10^5$ nodes, and the results are averaged over 10 simplicial complex realizations.

where $A_{q(\delta)}$ is the integral over all arrival times greater than $t_{j_q(\delta)}$, $X_{q(r),q(r+1)}$ is the integral over arrival times between $t_{j_{q(r)}}$ and $t_{j_{q(r+1)}}$, and $B_{q(0)}$ is the integral over arrival times less than $t_{j_{q(0)}}$.

All of these quantities can be expressed in term of the function $I^n_{\tau,t}$ given by

$$I^n_{\tau,t}=\int_\tau^t dt_n t_n^{\frac{1}{d+s}-1}\int_\tau^{t_n} dt_{n-1}t_{n-1}^{\frac{1}{d+s}-1}\cdots\int_\tau^{t_2}dt_1 t_1^{\frac{1}{d+s}-1}.$$

In particular, by distinguishing between the cases in which there is at least one node whose arrival time is being integrated over, and the case in which the allocation of positions specified by $\{q\}$ implies that there are no arrival times to integrate over, we obtain

$$A_{q(\delta)}=\begin{cases}\int_{t_{i_\delta}}^t dt_{j_d}t_{j_d}^{-\lambda-\frac{d}{d+s}}I^{d-q(\delta)-1}_{t_{i_\delta},t_{j_d}} & \text{if } 0\leqslant q(\delta)\leqslant d-1,\\ t_{i_\delta}^{-\lambda+\frac{s-1}{d+s}} & \text{if } \quad q(\delta)=d,\end{cases} \tag{51}$$

$$X_{q(r),q(r+1)}=\begin{cases}I^{q(r+1)-q(r)-1}_{t_{i_r},t_{i_{r+1}}} & \text{if } q(r+1)-q(r)>1,\\ 1 & \text{if } q(r+1)-q(r)=1,\end{cases} \tag{52}$$

$$B_{q(0)}=\begin{cases}I^{q(0)}_{0,t_{i_0}} & \text{if } \quad q(0)>0,\\ 1 & \text{if } \quad q(0)=0.\end{cases} \tag{53}$$

We note here that Eq. (50) may be simplified further by substituting the expression for p_α given in Eq. (38):

$$s_{d,\delta}(t,t_\alpha)=w_0\frac{(d-\delta)!}{(d+s)^{d-\delta}}t^\lambda t_{i_\delta}^{-\frac{d+s-\delta-1}{d+s}}\times\sum_{\{q\}}A_{q(\delta)}\left(\prod_{r=0}^{\delta-1}X_{q(r),q(r+1)}\right)B_{q(0)}. \tag{54}$$

This expression can be shown to depend only on the ratio between the time t and the time $t_\alpha=t_{i_\delta}$. Specifically, it can be shown (see Appendix A for details of the derivation) that $s_{d,\delta}(t,t_\alpha)$ is given by

$$s_{d,\delta}(t,t_\alpha) = \begin{cases} w_0 \frac{d-\delta}{(d+s)(\lambda_\delta-\lambda)}\left(\frac{t}{t_\alpha}\right)^{\lambda_\delta} + w_0\left[1 - \frac{d-\delta}{(d+s)(\lambda_\delta-\lambda)}\right]\left(\frac{t}{t_\alpha}\right)^{\lambda} & \text{if} \quad \lambda \neq \lambda_\delta, \\ w_0\left(\frac{t}{t_\alpha}\right)^{\lambda}\left[1 + \frac{d-\delta}{d+s}\log\left(\frac{t}{t_\alpha}\right)\right] & \text{if} \quad \lambda = \lambda_\delta, \end{cases} \tag{55}$$

where λ_δ is given by Eq. (29) and λ is given by Eq. (42).

For $t/t_\alpha \gg 1$ keeping only the leading terms of the above expression, we get

$$s_{d,\delta}(t,t_\alpha) \propto \begin{cases} \left(\frac{t}{t_\alpha}\right)^{\lambda_\delta} & \text{if} \quad \lambda < \lambda_\delta, \\ \left(\frac{t}{t_\alpha}\right)^{\lambda} & \text{if} \quad \lambda > \lambda_\delta, \\ \left(\frac{t}{t_\alpha}\right)^{\lambda}\log\left(\frac{t}{t_\alpha}\right) & \text{if} \quad \lambda = \lambda_\delta. \end{cases}$$

Finally, we can evaluate the scaling of the average generalized strength versus the generalized degree $s_{d,\delta}(k_{d,\delta})$ using the mean-field approximation. Toward that end, we keep only the leading terms for $t/t_\alpha \gg 1$ both in Eq. (28) for the average generalized degrees $k_{d,\delta}(t,t_\alpha)$ and in Eq. (55) for the average generalized strengths $s_{d,\delta}(t,t_\alpha)$, and we neglect the fluctuations of the generalized degrees $[k_{d,\delta}(t,t_\alpha) \simeq k^\alpha_{d,\delta}]$ and generalized strengths $[s_{d,\delta}(t,t_\alpha) \simeq s^\alpha_{d,\delta}]$. As long as $\lambda_\delta > 0$, we obtain

$$s_{d,\delta}(k_{d,\delta}) \propto \begin{cases} k_{d,\delta} & \text{for} \quad \lambda < \lambda_\delta, \\ k_{d,\delta}\ln k_{d,\delta} & \text{for} \quad \lambda = \lambda_\delta, \\ (k_{d,\delta})^{\lambda/\lambda_\delta} & \text{for} \quad \lambda > \lambda_\delta. \end{cases} \tag{56}$$

For $\lambda_\delta = 0$, instead we derive an exponential scaling of the average of the generalized strength versus the average of the generalized degree of the δ-faces, i.e.,

$$s_{d,\delta}(k_{d,\delta}) \propto e^{\beta k_{d,\delta}}, \tag{57}$$

with $\beta = \lambda\frac{d+s}{(1-s)c_\delta}$. These results predict that by tuning the parameter values m and m' [determining λ as for Eq. (42)], is possible to observe either linear, superlinear, or even exponential scaling of the generalized strengths versus the generalized degrees.

We stress here that the scaling relations Eqs. (56) and (57) are obtained in the limit $t/t_\alpha \gg 1$ neglecting the fluctuations of the generalized degrees and the generalized strengths over different network realizations. Therefore, these expressions need to be compared to numerical simulations to assess the limits of the considered approximations.

V. NUMERICAL SIMULATIONS

To check the validity of our mean-field calculations, we have run extensive simulations of the model. When writing the program to implement the model, care should be taken in efficiently storing the information about the composition of the simplicial complex. In fact, for a simplicial complex of N nodes the total number of potential δ'-simplices scales like $N^{\delta'+1}$, therefore maintaining adjacency tensors with entries for every potential δ'-simplex would require storing $N^{\delta'+1}$ integers. Since in our model the actual number of d-simplices and δ-faces scales linearly with N, we can handle efficiently the information about the simplicial complex structure by keeping a list of the simplices and the faces generated by the model rather than maintaining adjacency tensors.

Here we report and discuss in particular the simulation results obtained for simplicial complexes of dimension $d = 3$ with all the possible values of the flavor $s = -1,0,1$ and a variety of choices of m and m'. Our main goal is to characterize the limit of validity of the mean-field calculations performed in the previous section.

In Fig. 3, we report the simulation results for the generalized degree distribution $P_{d,\delta}(k_{d,\delta})$ and $N = 10^5$ averaged over 10 realizations of the model. We observe that the mean-field calculation predicts exactly for which dimension δ and for which flavor s we observe binomial, exponential, or power-law distribution.

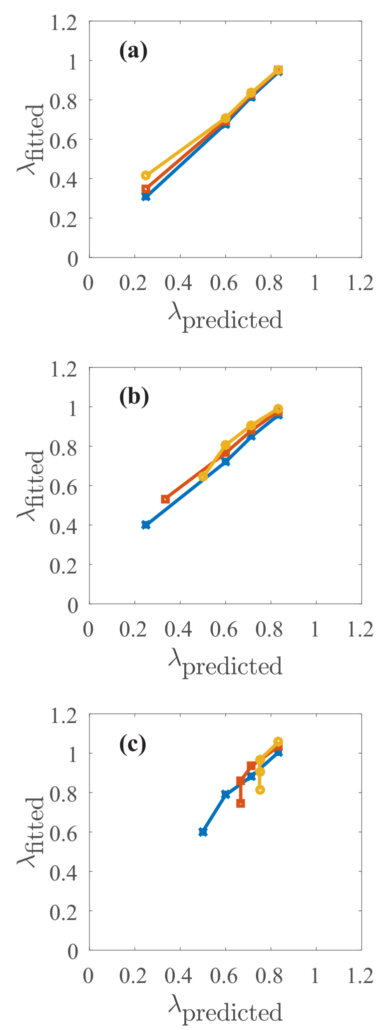

FIG. 5. The exponents λ_{fitted} obtained by fitting Eq. (58) to the data in Fig. 4 are shown vs the predicted exponents $\lambda_{\text{predicted}}$ given by Eq. (59) for different δ-faces. The panels (a), (b), and (c) refer, respectively, to triangles ($\delta = 2$), links ($\delta = 1$), and nodes ($\delta = 0$). The blue stars, red squares, and yellow circles indicate the data obtained, respectively, for the the flavors $s = -1$, 0, and 1.

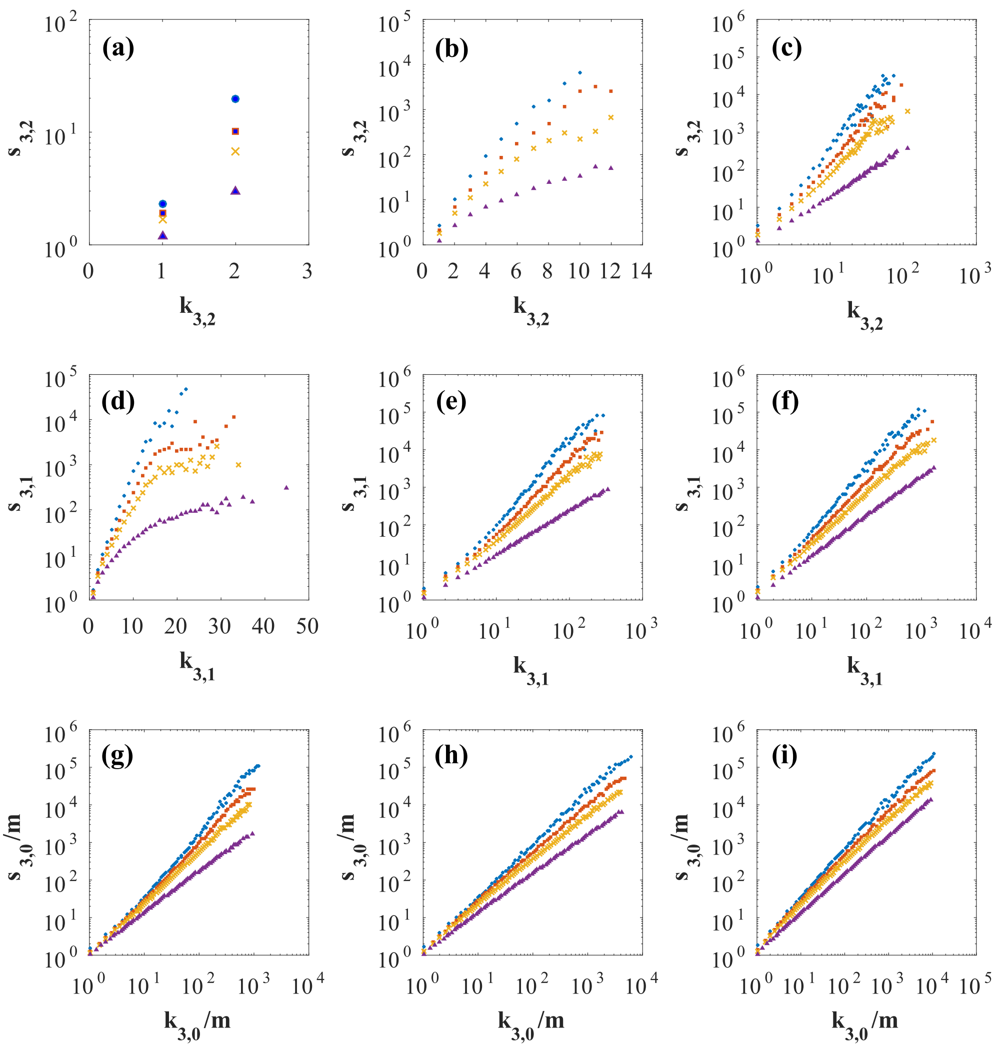

FIG. 6. The average generalized strengths of δ-faces as a function of their corresponding generalized degree $s_{d,\delta}(k_{d,\delta})$ are shown for simplicial complexes of dimension $d = 3$ and flavor $s = -1$ [panels (a), (d), and (g)], $s = 0$ [panels (b), (e), and (h)], and $s = 1$ [panels (c), (f), and (i)] and for faces of dimension $\delta = 2$ [panels (a), (b), and (c)], $\delta = 1$ [panels (d), (e), and (f)], and $\delta = 0$ [panels (g), (h), and (i)]. The results of simulations are shown for $(m = 1, m' = 5)$, $(m = 2, m' = 5)$, $(m = 2, m' = 3)$, and $(m = 3, m' = 1)$ (blue circles, red squares, yellow crosses, and purple triangles, respectively). The simulated simplicial complexes have $N = 10^5$ nodes, and the results are averaged over 10 simplicial complex realizations.

In Fig. 4 we display the average generalized strengths $s_{d,\delta}(t,t_\alpha)$ of δ-faces α as a function of their arrival time t_α. We observe a clear power-law scaling of $s_{d,\delta}(t,t_\alpha)$ as a function of t/t_α for $t/t_\alpha \gg 1$ as predicted by the mean-field approximation [Eq. (56)]. To evaluate more in detail the limits of validity of the mean-field equations, we performed the power-law fits

$$s_{d,\delta}(t,t_\alpha) = a\left(\frac{t}{t_\alpha}\right)^{\lambda_{\text{fitted}}}, \tag{58}$$

valid for $t/t_\alpha \gg 1$, and we compared the fitted exponent λ_{fitted} with the predicted exponent obtained from Eq. (56),

$$\lambda_{\text{predicted}} \simeq \max(\lambda,\lambda_\delta). \tag{59}$$

The comparison between the exponents λ_{fitted} and $\lambda_{\text{predicted}}$ is shown in Fig. 5 for simplicial complexes of dimension $d = 3$ and different values of m and m' determining λ. We observe that while the overall trend of λ_{fitted} is captured by the mean-field result, some deviations are observed. These deviations become more significant for $\lambda \simeq \lambda_\delta$, where it is expected to be more difficult to observe the leading term in Eq. (55) starting for finite-time simulations results.

Finally, as discussed in Sec. IV D, for simplicial complexes with a large number of nodes, the mean-field approximation predicts that the generalized strengths of the faces are related to their generalized degrees by the scaling relation given in Eqs. (56) and (57). Nevertheless, we have discussed that this approximation neglects the role of fluctuations for the generalized degree and for the generalized strengths. Therefore, it is important to check to what extend the mean-field calculations capture the simulation results. In Fig. 6, we show the average generalized strength versus the generalized degree $s_{d,\delta}(k_{d,\delta})$. We observe that the role of fluctuations is particularly pronounced for δ-faces with exponential generalized degree distributions (i.e., $\delta = d - 2$ for $s = -1$ and $\delta = d - 1$ for $s = 0$). These fluctuations are more significant for values of the parameters m and m' corresponding to low values of λ.

Instead, we observe that for the other δ-faces the mean-field predictions provide a rather good prediction of the scaling of the average generalized strength $s_{d,\delta}(k_{d,\delta})$.

VI. CONCLUSIONS

In this paper, we have presented a nonequilibrium model for weighted simplicial complexes. In this model, simplicial complexes evolve at each time (A) by the addition of a new node belonging to m new d-dimensional simplices, and (B) by the reinforcement of the weights of m' d-dimensional simplices.

The model generates simplicial complexes with nontrivial topology, including manifolds, with either constant Euler characteristics (for $m = 1$) or with continuously decreasing Euler characteristics. The skeleton of these simplicial complexes is a network with notable complex structure, including a high clustering coefficient (ensured by the simplicial complex structure) and heterogeneous scale-free degree distribution for $d \geqslant 3$.

Here we have focused on the rich interplay between topological properties of the simplicial complexes and the distribution of the weights of the simplices. We have found that is possible to extend the strength versus degree analysis performed in simple networks to simplicial complexes by characterizing the functional relation between the generalized strengths of their faces versus the corresponding generalized degrees. Specifically, the proposed model is able to generate simplicial complexes where the generalized strength grows linearly, superlinearly, or exponentially as a function of the parameters m and m' of the model.

We believe that this model could be rather fruitful for modeling real-world simplicial complexes such as collaboration networks that are typically weighted. Additionally, the model could be used as a benchmark to test the wide range of topological and geometrical measures and computational techniques that have been proposed in recent years for the study of real datasets. These include different definitions of curvature, and the persistent homology conducted using a filtration based on the weights of the links or on the simplices.

APPENDIX A: MAIN STEPS OF THE DERIVATION OF EQ. (55)

In the main body of the paper, we have derived the following equation [Eq. (54)] for the average generalized strengths $s_{d,\delta}(t,t_\alpha)$ of the δ-face α with arrival time t_α:

$$s_{d,\delta}(t,t_\alpha) = w_0 \frac{(d-\delta)!}{(d+s)^{d-\delta}} t^{\lambda} t_{i_\delta}^{-\frac{d+s-\delta-1}{d+s}} \times \sum_{\{q\}} A_{q(\delta)} \left(\prod_{r=0}^{\delta-1} X_{q(r),q(r+1)} \right) B_{q(0)}, \quad \text{(A1)}$$

where $A_{q(\delta)}$, $B_{q(\delta)}$, and $X_{q(r),q(r+1)}$ are defined, respectively, by Eqs. (A7), (53), and (52) of the main text.

To obtain explicit expressions for $A_{q(\delta)}$, $X_{q(r),q(r+1)}$, and $B_{q(0)}$, let us observe that the function $I^n_{\tau,t}$ defined as

$$I^n_{\tau,t} = \int_\tau^t dt_n t_n^{\frac{1}{d+s}-1} \int_\tau^{t_n} dt_{n-1} t_{n-1}^{\frac{1}{d+s}-1} \cdots \int_\tau^{t_2} dt_1 t_1^{\frac{1}{d+s}-1} \quad \text{(A2)}$$

can also be written as

$$I^n_{\tau,t} = \frac{(d+s)^n}{n!} \left(t^{\frac{1}{d+s}} - \tau^{\frac{1}{d+s}} \right)^n = \frac{(d+s)^n}{n!} \sum_{r=0}^{n} \binom{n}{r} (-1)^r t^{\frac{n-r}{d+s}} \tau^{\frac{r}{d+s}}. \quad \text{(A3)}$$

This result allows us to express $X_{q(r),q(r+1)}$ as

$$X_{q(r),q(r+1)} = \begin{cases} I^{q(r+1)-q(r)-1}_{t_{i_r},t_{i_{r+1}}} & \text{if} \quad q(r+1)-q(r) > 1, \\ 1 & \text{if} \quad q(r+1)-q(r) = 1, \end{cases} = \begin{cases} \frac{(d+s)^{q(r+1)-q(r)-1}}{[q(r+1)-q(r)-1]!} \left(t_{i_{r+1}}^{\frac{1}{d+s}} - t_{i_r}^{\frac{1}{d+s}} \right)^{q(r+1)-q(r)-1} & \text{if} \quad q(r+1)-q(r) > 1, \\ 1 & \text{if} \quad q(r+1)-q(r) = 1. \end{cases} \quad \text{(A4)}$$

Similarly, $B_{q(0)}$ can be expressed as

$$B_{q(0)} = \begin{cases} I^{q(0)}_{0,t_{i_0}} & \text{if} \quad q(0) > 0, \\ 1 & \text{if} \quad q(0) = 0, \end{cases} = \begin{cases} \frac{(d+s)^{q(0)}}{q(0)!} t_{i_0}^{\frac{q(0)}{d+s}} & \text{if} \quad q(0) > 0, \\ 1 & \text{if} \quad q(0) = 0. \end{cases} \quad \text{(A5)}$$

Finally, using Eq. (A3) and the definition of $A_{q(\delta)}$ that we rewrite here for convenience,

$$A_{q(\delta)} = \begin{cases} \int_{t_{i_\delta}}^t dt_{j_d} t_{j_d}^{-\lambda-\frac{d}{d+s}} I^{d-q(\delta)-1}_{t_{i_\delta},t_{j_d}} & \text{if} \quad 0 \leqslant q(\delta) \leqslant d-1, \\ t_{i_\delta}^{-\lambda+\frac{s-1}{d+s}} & \text{if} \quad q(\delta) = d, \end{cases} \quad \text{(A6)}$$

we obtain

$$A_{q(\delta)} = \begin{cases} \frac{(d+s)^{d-q(\delta)-1}}{[d-q(\delta)-1]!} \sum_{r=0}^{d-q(\delta)-1} \binom{d-q(\delta)-1}{r} (-1)^r t_{i_\delta}^{\frac{r}{d+s}} \int_{t_{i_\delta}}^{t} dt_{j_d} t_{j_d}^{-\lambda+\frac{d+s-q(\delta)-r-1}{d+s}-1} & \text{if} \quad 0 \leqslant q(\delta) \leqslant d-1, \\ t_{i_\delta}^{-\lambda+\frac{s-1}{d+s}} & \text{if} \quad q(\delta)=d. \end{cases} \tag{A7}$$

In the case $q(\delta) \leqslant d-1$, the integral present in Eq. (A7) has two separate expressions for $\lambda = \frac{d+s-q(\delta)-1-r}{d+s}$ and $\lambda \neq \frac{d+s-q(\delta)-1-r}{d+s}$. By performing the integral, we find the following expression for $A_{q(\delta)}$:

$$A_{q(\delta)} = \begin{cases} (d+s)^{d-q(\delta)-1} \sum_{r=0}^{d-q(\delta)-1} D_{q(\delta)+r} t_{i_\delta}^{\frac{r}{d+s}} \frac{(-1)^r}{r!} & \text{if } 0 \leqslant q(\delta) \leqslant d-1, \\ t_{i_\delta}^{-\lambda+\frac{s-1}{d+s}} & \text{if } q(\delta)=d, \end{cases} \tag{A8}$$

where the quantities $D_{q(\delta)+r}$ depend on $q(\delta)$ and r only through their sum, and they are given by

$$D_{q(\delta)+r} = \begin{cases} \frac{1}{[d-q(\delta)-r-1]!} \left(\frac{d+s-q(\delta)-1-r}{d+s} - \lambda\right)^{-1} \left(t^{\frac{d+s-q(\delta)-1-r}{d+s}-\lambda} - t_{i_\delta}^{\frac{d+s-q(\delta)-1-r}{d+s}-\lambda}\right) & \text{if} \quad \lambda \neq \frac{d+s-q(\delta)-1-r}{d+s}, \\ \frac{1}{[d-q(\delta)-r-1]!} \log\left(\frac{t}{t_{i_\delta}}\right) & \text{if} \quad \lambda = \frac{d+s-q(\delta)-1-r}{d+s}. \end{cases} \tag{A9}$$

To calculate the generalized strengths given by Eq. (A1), let us observe that

$$\sum_{\{q\}} A_{q(\delta)} \left(\prod_{r=0}^{\delta-1} X_{q(r),q(r+1)} \right) B_{q(0)} = \sum_{q(\delta)=\delta}^{d} A_{q(\delta)} R_{q(\delta),\delta}, \tag{A10}$$

where $R_{q(r),r}$ are functions defined recursively by the following pair of equations:

$$R_{q(1),1} = \sum_{q(0)=0}^{q(1)-1} X_{q(0),q(1)} B_{q(0)}, \tag{A11}$$

$$R_{q(\beta),\beta} = \sum_{q(\beta-1)=\beta-1}^{q(\beta)-1} X_{q(\beta-1),q(\beta)} R_{q(\beta-1),\beta-1}. \tag{A12}$$

The solution of these equations (see the following appendix for details of this derivation) reads

$$R_{q(\beta),\beta} = (d+s)^{q(\beta)-\beta} \frac{t_{i_\beta}^{\frac{q(\beta)-\beta}{d+s}}}{[q(\beta)-\beta]!}. \tag{A13}$$

Therefore, the average generalized strength $s_{d,\delta}(t,t_\alpha)$ may be written as

$$s_{d,\delta}(t,t_\alpha) = w_0 \frac{(d-\delta)!}{(d+s)^{d-\delta}} t^\lambda t_{i_\delta}^{-\frac{d+s-\delta-1}{d+s}} \left[A_d R_{d,\delta} + \sum_{q(\delta)=\delta}^{d-1} A_{q(\delta)} R_{q(\delta)} \right]. \tag{A14}$$

Using Eqs. (A9) and Eq. (A13), we get

$$\begin{aligned} \sum_{q(\delta)=\delta}^{d-1} A_{q(\delta)} R_{q(\delta),\delta} &= (d+s)^{d-\delta-1} \sum_{q(\delta)=\delta}^{d-1} \sum_{r=0}^{d-q(\delta)-1} D_{q(\delta)+r} t_{i_\delta}^{\frac{q(\delta)+r-\delta}{d+s}} \frac{(-1)^r}{r![q(\delta)-\delta]!} \\ &= (d+s)^{d-\delta-1} D_{q(\delta)+r} t_{i_\delta}^{\frac{q(\delta)+r-\delta}{d+s}} \Big|_{q(\delta)+r=\delta} = (d+s)^{d-\delta-1} D_\delta. \end{aligned} \tag{A15}$$

Note that in deriving Eq. (A15), we have used the following mathematical identity:

$$\sum_{x=a}^{b} \sum_{y=0}^{b-x} f(x+y) \frac{(-1)^y}{y!(x-a)!} = f(a), \tag{A16}$$

valid for integers $a,b > 0$ with $a < b$. Therefore, the average generalized strength given by Eq. (A14) can be written as

$$\begin{aligned} s_{d,\delta}(t,t_\alpha) &= w_0 \frac{(d-\delta)!}{(d+s)^{d-\delta}} t^\lambda t_{i_\delta}^{-\frac{d+s-\delta-1}{d+s}} \left[A_d R_{d,\delta} + \sum_{q(\delta)=\delta}^{d-1} A_{q(\delta)} R_{q(\delta)} \right] \\ &= w_0 \frac{(d-\delta)!}{(d+s)^{d-\delta}} t^\lambda t_{i_\delta}^{-\frac{d+s-\delta-1}{d+s}} \left[\frac{(d+s)^{d-\delta}}{(d-\delta)!} t_{i_\delta}^{-\lambda+\frac{d+s-\delta-1}{d+s}} + (d+s)^{d-\delta-1} D_\delta \right], \end{aligned} \tag{A17}$$

which simplifies to

$$s_{d,\delta}(t,t_\alpha) = w_0\left(\frac{t}{t_{i_\delta}}\right)^{\lambda} + w_0\frac{(d-\delta)!}{d+s}t^{\lambda}t_{i_\delta}^{-\frac{d+s-\delta-1}{d+s}}D_\delta. \tag{A18}$$

As noted earlier, D_δ takes different forms in the cases $\lambda \neq \lambda_\delta = \frac{d+s-\delta-1}{d+s}$ and $\lambda = \lambda_\delta = \frac{d+s-\delta-1}{d+s}$. Inserting Eq. (A9) into Eq. (A18) leads to our final expression for the generalized strength:

$$s_{d,\delta}^{\alpha}(t) = \begin{cases} w_0\frac{d-\delta}{(d+s)(\lambda_\delta-\lambda)}\left(\frac{t}{t_{i_\delta}}\right)^{\lambda_\delta} + w_0\left[1-\frac{d-\delta}{(d+s)(\lambda_\delta-\lambda)}\right]\left(\frac{t}{t_{i_\delta}}\right)^{\lambda} & \text{if} \quad \lambda\neq\lambda_\delta, \\ w_0\left(\frac{t}{t_{i_\delta}}\right)^{\lambda}\left[1+\frac{d-\delta}{d+s}\log\left(\frac{t}{t_{i_\delta}}\right)\right] & \text{if} \quad \lambda=\lambda_\delta. \end{cases} \tag{A19}$$

Since this equation is the same as Eq. (55) of the main text, this concludes our discussion here.

APPENDIX B: DERIVATION OF EQ. (A13)

In this appendix, our goal is to show that Eq. (A13) holds. This equation is given by

$$R_{q(\beta),\beta} = (d+s)^{q(\beta)-\beta}\frac{t_{i_\beta}^{\frac{q(\beta)-\beta}{d+s}}}{[q(\beta)-\beta]!}, \tag{B1}$$

where $R_{q(r),r}$ are functions defined recursively by the following pair of equations:

$$R_{q(1),1} = \sum_{q(0)=0}^{q(1)-1} X_{q(0),q(1)}B_{q(0)}, \tag{B2}$$

$$R_{q(\beta),\beta} = \sum_{q(\beta-1)=\beta-1}^{q(\beta)-1} X_{q(\beta-1),q(\beta)}R_{q(\beta-1),\beta-1}. \tag{B3}$$

Toward that end, we first check that (B1) holds for $\beta = 1$. Inserting Eq. (A4) for $X_{q(0),q(1)}$ and Eq. (A5) for $B_{q(0)}$ into Eq. (B2) gives

$$R_{q(1),1} = \sum_{q(0)=0}^{q(1)-1}\sum_{l_0=0}^{q(1)-q(0)-1}\frac{(d+s)^{q(1)-1}(-1)^{l_0}}{[q(1)-q(0)-1-l_0]!(l_0)!q(0)!} \times t_{i_0}^{\frac{q(0)+l_0}{d+s}}t_{i_1}^{\frac{q(1)-q(0)-1-l_0}{d+s}}. \tag{B4}$$

We note that the expression being summed over factorizes into a term depending on $q(0)$ and l_0 only through their sum and a term depending on $q(0)$ and l_0 otherwise:

$$R_{q(1),1} = \sum_{q(0)=0}^{q(1)-1}\sum_{l_0=0}^{q(1)-q(0)-1} f(q(0)+l_0)\frac{(-1)^{l_0}}{l_0!q(0)!}, \tag{B5}$$

where

$$f(q(0)+l_0) = (d+s)^{q(1)-1}\frac{t_{i_0}^{\frac{q(0)+l_0}{d+s}}t_{i_1}^{\frac{q(1)-q(0)-1-l_0}{d+s}}}{[q(1)-q(0)-1-l_0]!}. \tag{B6}$$

Using the mathematical identity Eq. (A16), Eq. (B5) simplifies to

$$R_{q(1),1} = f(0) = (d+s)^{q(1)-1}\frac{t_{i_1}^{\frac{q(1)-1}{d+s}}}{[q(1)-1]!}. \tag{B7}$$

So (B1) holds in the case $\beta = 1$.

We now show that in general if Eq. (B1) holds for some β, then it must also hold for $\beta+1$. Substituting Eq. (A4) and Eq. (B1) into Eq. (B3) gives

$$R_{q(\beta+1),\beta+1} = \sum_{q(\beta)=\beta}^{q(\beta+1)-1}\sum_{l_\beta=0}^{q(\beta+1)-q(\beta)-1} \times\left[\frac{(d+s)^{q(\beta+1)-\beta-1}(-1)^{l_\beta}}{[q(\beta+1)-q(\beta)-1-l_\beta]!(l_\beta)![q(\beta)-\beta]!} \times t_{i_\beta}^{\frac{q(\beta)+l_\beta-\beta}{d+s}}t_{i_{\beta+1}}^{\frac{q(\beta+1)-q(\beta)-1-l_\beta}{d+s}}\right]. \tag{B8}$$

Similar to the $\beta = 1$ case, we may write (B8) in the form

$$R_{q(\beta+1),\beta+1} = \sum_{q(\beta)=\beta}^{q(\beta+1)-1}\sum_{l_\beta=0}^{q(\beta+1)-q(\beta)-1} f(q(\beta)+l_\beta)\frac{(-1)^{l_\beta}}{l_\beta!q(\beta)!}, \tag{B9}$$

where in this case the term depending only on $q(\beta)$ and l_β through the sum of the two is

$$f(q(\beta)+l_\beta) = (d+s)^{q(\beta+1)-\beta-1}\frac{t_{i_\beta}^{\frac{q(\beta)+l_\beta-\beta}{d+s}}t_{i_{\beta+1}}^{\frac{q(\beta+1)-q(\beta)-1-l_\beta}{d+s}}}{[q(\beta+1)-q(\beta)-1-l_\beta]!}. \tag{B10}$$

Using the identity (A16) allows us to make the simplification

$$R_{q(\beta+1),\beta+1} = f(\beta) = (d+s)^{q(\beta+1)-\beta-1} \times\frac{t_{i_{\beta+1}}^{\frac{q(\beta+1)-\beta-1}{d+s}}}{[q(\beta+1)-\beta-1]!}, \tag{B11}$$

which confirms Eq. (B1), or equivalently Eq. (A13).

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