Supplementary Information (SI)

1 Mapping to a thermal disordered system

We assume the starting equation to be

$$\frac{dN_i}{dt} = \frac{r_i}{K_i} N_i \left[f_i(N_i) + \xi_i - \sum_{j(\neq i)} \alpha_{ij} N_j \right] + \sqrt{N_i}\eta_i + \lambda_i \tag{1}$$

with $\alpha_{i,j}$ a symmetric matrix, and η_i a white noise: $\langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta(t-t')$. Lotka-Volterra (LV) classical system of equations for interacting species can be obtained by setting $f_i(N_i) = K_i - N_i$.

By scaling the variables N_i by $\langle K_i \rangle = \mu_K$: $\widetilde{N_i} = N_i/\mu_k$ and consequently also the relevant parameters $\widetilde{K_i} = K_i/\mu_K$, $\widetilde{\lambda_i} = \lambda_i/\mu_K$, $\widetilde{T} = T/\mu_K$, $\widetilde{\xi_i} = \xi_i/\mu_K$, and also the function $\widetilde{f_i}(\widetilde{N_i}) = f_i(N_i)/\mu_K$ (which is true in the LV case) we get an identical dynamical equation in terms of the new variables and the old α_{ij}, and r_i. Without loss of generality then we set $\mu_K = 1$ and forget about all the tildas.

We also prefer to define an interaction matrix $\theta_{ij} = r_i\alpha_{ij}/K_i = \rho\alpha_{ij}$. In doing this we assume that the ratio $\rho = r_i/K_i$ is i-independent so that α_{ij} and θ_{ij} are symmetric at the same time.

Finally we define the function $V(N_i)$ such that $-\nabla_{N_i} V_i(N_i) = \rho f_i(N_i)$ (in the LV case $V_i(N_i) = -\rho(K_i N_i - N_i^2/2)$), we set $\xi_i = 0$ and $\lambda_i = \lambda$.

The dynamical equations then read

$$\frac{dN_i}{dt} = -N_i \nabla_{N_i} V_i(N_i) - N_i \sum_{j(\neq i)} \theta_{ij} N_j + \sqrt{N_i}\eta_i + \lambda \ . \tag{2}$$

We want to show that this equation admits an invariant probability distribution in terms of a Hamiltonian H. To do so we derive the corresponding Fokker-Planck equation[1]. We consider a generic observable $O(\{N_j\})$ and the time derivative of its average over the thermal noise $\frac{d}{dt}\langle O(\{N_j\})\rangle$. This derivative will obtain us the time derivative of the distribution of our variables originated by the thermal noise itself:

$$\begin{aligned} \frac{d}{dt}\langle O(\{N_j\})\rangle &= \frac{d}{dt}\int \prod_i dN_i P(\{N_j\},t) O(\{N_j\}) \\ &= \int \prod_i dN_i \frac{\partial}{\partial t} P(\{N_j\},t) O(\{N_j\}) \ . \end{aligned} \tag{3}$$

Adopting the Ito convention[1], the LHS of the equation (3) corresponds to

$$\frac{d}{dt}\langle O(\{N_j\})\rangle = \langle \sum_i \frac{\partial O}{\partial N_i}\frac{dN_i}{dt}\rangle + \frac{1}{2}\langle \sum_{i,j} \frac{\partial^2 O}{\partial N_i \partial N_j}\eta_i\eta_j\sqrt{N_i N_j}\rangle \ . \tag{4}$$

In the Ito prescription variables are not correlated with noise at the same time so, also using equation (2), the previous equation becomes

$$\frac{d}{dt}\langle O(\{N_j\})\rangle = \langle \sum_i \frac{\partial O}{\partial N_i}\mathcal{D}(\{N_j\})\rangle + T\langle \sum_i \frac{\partial^2 O}{\partial N_i^2} N_i\rangle \ , \tag{5}$$

where

$$\mathcal{D}(\{N_j\}) = -N_i \nabla_{N_i} V_i(N_i) - N_i \sum_{j(\neq i)} \theta_{ij} N_j + \lambda \ .$$

Re-writing now the average over the noise as an average over $P(\{N_j\}, t)$

$$\frac{d}{dt}\langle O(\{N_j\})\rangle = \int \prod_i dN_i P(\{N_j\}, t) \times \qquad (6)$$
$$\sum_i \left(\frac{\partial O}{\partial N_i}\mathcal{D}(\{N_j\}) + T\frac{\partial^2 O}{\partial N_i^2} N_i \right) ,$$

and integrating by parts we get

$$\tfrac{d}{dt}\langle O(\{N_j\})\rangle = \int \prod_i dN_i O(\{N_j\}) \times \qquad (7)$$
$$\sum_i \left\{ T\frac{\partial^2}{\partial N_i^2}[P(\{N_j\}, t)N_i] - \frac{\partial}{\partial N_i}[\mathcal{D}(\{N_j\})P(\{N_j\}, t)] \right\} \ .$$

By comparison with equation (3) we can now write the Fokker-Planck equation as

$$\frac{\partial}{\partial t}P(\{N_j\}, t) = \qquad (8)$$
$$\sum_i \left\{ T\frac{\partial^2}{\partial N_i^2}[P(\{N_j\}, t)N_i] - \frac{\partial}{\partial N_i}[\mathcal{D}(\{N_j\})P(\{N_j\}, t)] \right\} \ .$$

The equilibrium distribution must satisfy then

$$0 = \sum_i \left\{ T\frac{\partial^2}{\partial N_i^2}[P(\{N_j\}, t)N_i] - \frac{\partial}{\partial N_i}[\mathcal{D}(\{N_j\})P(\{N_j\}, t)] \right\} \ ,$$

which is obtained by imposing $\forall i$

$$\frac{\partial P(\{N_j\}, t)}{\partial N_i} = \left[\frac{\mathcal{D}(\{N_j\})}{T} - 1 \right] \frac{P(\{N_j\}, t)}{N_i} \ . \qquad (9)$$

By asking that this equilibrium distribution is also of the form

$$P = Z^{-1} \exp(-\beta H) \ , \qquad (10)$$

with Z the usual normalizing partition function, we obtain

$$\frac{\partial P}{\partial N_i} = -P\beta \frac{\partial H}{\partial N_i} \qquad (11)$$

and hence that

$$\frac{\partial H}{\partial N_i} = -\frac{1}{N_i}[\mathcal{D}(\{N_j\}) - T] \qquad (12)$$

so finally

$$H = \sum_i V_i(N_i) + \sum_{(i,j)} \theta_{ij} N_i N_j + (T - \lambda) \sum_i \ln(N_i) \ . \qquad (13)$$

Note that having $\lambda > T$ (even when $T \to 0$) is a fundamental element to have a regularized $P(\{N_j\})$ at small N_j.

In conclusion the original dynamical equations describe the dynamical evolution of a system whose thermodynamics is determined by the Hamiltonian just obtained.

2 Replica computation

We use a replica approach to analyze the thermodynamics of the system characterized by the Hamiltonian (13). Recall that θ_{ij} are assumed to be Gaussian distributed with mean $\rho\mu/S$ and variance $\rho^2\sigma^2/S$. We evaluate the free-energy of the system by applying the replica trick to perform sample averages

$$-\beta F = \lim_{n\to 0} \frac{\ln \overline{Z^n}}{n} \ . \tag{14}$$

We then evaluate the replicated partition function as

$$\overline{Z^n} = \overline{\int \prod_{i,(ij)} dN_i^a d\theta_{ij} \exp\left(- \sum_{(ij)} \frac{(\theta_{ij} - \rho\mu/S)^2}{2\rho^2\sigma^2/S} - \beta H(\{N_i\}) \right)}^V \ , \tag{15}$$

where the average over the disorder contained in V remains to be done, and which, through standard replica manipulations[2], becomes

$$-\beta n F = \ln \overline{\int \prod_{a,a<b} dQ_{ab} dQ_{aa} dH_a \exp\left[S\mathcal{A}(Q_{a,b}, Q_{a,a}, H_a) \right]}^V \tag{16}$$

with

$$\begin{aligned} \mathcal{A}(Q_{a,b}, Q_{a,a}, H_a) = & -\rho^2\sigma^2\beta^2 \sum_{a<b} \frac{Q_{ab}^2}{2} - \rho^2\sigma^2\beta^2 \sum_a \frac{Q_{aa}^2}{4} \\ & + \rho\mu\beta \sum_a \frac{H_a^2}{2} + \frac{1}{S} \sum_i \ln Z_i \end{aligned} \tag{17}$$

In the last equation the effective on-site partition function $Z_i = \int \prod_a dN_i^a \exp\left(-\beta H_{eff}(\{N^a\}_i)\right)$ is obtained from an effective Hamiltonian

$$\begin{aligned} H_{eff}(\{N^a\}_i) = & -\beta\rho^2\sigma^2 \sum_{a<b} N_i^a N_i^b Q_{ab} - \beta\rho^2\sigma^2 \sum_a (N_i^a)^2 \frac{Q_{aa}}{2} \\ & + \sum_a \rho\mu N_i^a H_a + V_i(N_i^a) + (T-\lambda) \log N_i^a \end{aligned} \tag{18}$$

where the order parameters satisfy the self-consistent equations

$$Q_{ab} = \frac{1}{S} \sum_i \overline{\langle N_i^a N_i^b \rangle_{\text{AR}}}^V \ , \qquad H_a = \frac{1}{S} \sum_i \overline{\langle N_i^a \rangle_{\text{AR}}}^V$$

and the averages over configurations of all replicas (AR) are performed with the effective Hamiltonian H_{eff}.

3 Replica Symmetric Solution

Replica symmetry is correct when only a single equilibrium (or minimum) in the (free-)energy landscape governs the thermodynamic behavior. Formally, one assumes:

$$Q_{ab} = q_0 \, \forall a \neq b \qquad , \qquad Q_{aa} = q_D \, \forall a \qquad , \qquad H_a = h \, \forall a \ .$$

The free-energy expression becomes in this case

$$-\beta n F = \ln \overline{\int dq_0 dq_D dh \exp\left[S\mathcal{A}(q_0, q_1, h)\right]}^V \tag{19}$$

$$\mathcal{A}(q_0, q_1, h) = -\rho^2\sigma^2\beta^2\frac{n(n-1)}{4}q_0^2 - \rho^2\sigma^2\beta^2\frac{n}{4}q_D^2 \tag{20}$$

$$+\rho\mu\beta\frac{n}{2}h^2 + \frac{1}{S}\sum_i \ln Z_i \tag{21}$$

with an effective Hamiltonian for Z_i

$$H_{eff}(\{N^a\}_i) = -\frac{\beta\rho^2\sigma^2}{2}q_0\left(\sum_a N_i^a\right)^2 - \frac{\beta\rho^2\sigma^2}{2}(q_D - q_0)\sum_a N_i^{a\,2}$$

$$+\sum_a \rho\mu h N_i^a + V_i(N_i^a) + (T-\lambda)\log N_i^a \ . \tag{22}$$

To decouple replicas we exploit standard properties of Gaussian integrals and we get

$$Z_i = \int \frac{dz_i}{\sqrt{2\pi}} \exp\left[-\frac{z_i^2}{2}\right] \int \prod_a dN_i^a \exp\left[-\beta\sum_a H_{\mathrm{RS}}(N_i^a, z_i)\right]$$

with

$$H_{\mathrm{RS}}(N_i, z_i) = -\rho^2\sigma^2\beta(q_D - q_0)\frac{N_i^2}{2} + \left(\rho\mu h - z_i\rho\sqrt{q}_0\sigma\right)N_i$$

$$+V_i(N_i) + (T-\lambda)\log N_i \ . \tag{23}$$

By maximizing the action $\mathcal{S}$ at the exponent of (19)

$$\mathcal{S} = -\frac{n(n-1)}{4}\beta^2\rho^2\sigma^2 S q_0^2 - \frac{n}{4}\beta^2\rho^2\sigma^2 S q_D^2 + \frac{n}{2}\beta\rho\mu S h^2 + \sum_i \ln Z_i \tag{24}$$

we get the following saddle point (SP) equations on the introduced parameters

$$q_0 = \overline{\frac{1}{S}\sum_i \langle N_i^a N_i^b\rangle_{\mathrm{AR}}}^V \tag{25}$$

$$q_D = \overline{\frac{1}{S}\sum_i \langle N_i^{a\,2}\rangle_{\mathrm{AR}}}^V \tag{26}$$

$$h = \overline{\frac{1}{S}\sum_i \langle N_i^a\rangle_{\mathrm{AR}}}^V \tag{27}$$

with

$$\overline{\langle (N^b)^p (N^c)^r\rangle_{\mathrm{AR}}}^V =$$

$$= \overline{\frac{\int \frac{dz}{\sqrt{2\pi}}\prod_a dN^a \exp\left[-\frac{z^2}{2} - \beta\sum_a H_{\mathrm{RS}}(N^a, z)\right](N^b)^p(N^c)^r}{\int \frac{dz}{\sqrt{2\pi}}\prod_a dN^a \exp\left[-\frac{z^2}{2} - \beta\sum_a H_{\mathrm{RS}}(N^a, z)\right]}}^V$$

where b, c denotes replica indices and p, r, are the powers of the abundance we are considering. In the $n \to 0$ limit the formula above can be expressed in terms of thermal averages $\langle\cdot\rangle_{\rm 1R}$ over single species and single replica with Hamiltonian H_{RS}, and the disorder average $\overline{\cdot}$ representing the average over the disorder contained in $V(N)$ and the Gaussian integral over z with mean zero and unit variance

$$\overline{\langle (N^a)^p (N^b)^r \rangle_{\rm AR}}^V = \overline{\langle N^p\rangle_{\rm 1R}\langle N^r\rangle_{\rm 1R}} \tag{28}$$

where

$$\langle\ \cdot\ \rangle_{\rm 1R} = \frac{\int dN \exp\left[-\beta H_{\rm RS}(N,z)\right]\ \cdot}{\int dN \exp\left[-\beta H_{\rm RS}(N,z)\right]}\ . \tag{29}$$

Hence we can write

$$q_0 = \overline{\langle N\rangle^2_{\rm 1R}} \tag{30}$$

$$q_D = \overline{\langle N^2\rangle_{\rm 1R}} \tag{31}$$

$$h = \overline{\langle N\rangle_{\rm 1R}}\ . \tag{32}$$

The average over a single replica coincides with the standard average first over the Boltzmann weight and then over the quenched disorder. It is possible to reduce the latter to the former because the model is mean-field. The resulting equations have a clear interpretation: each species is subjected to its own potential V and two extra terms due to the overall mean-field interaction with the rest of the system. Two non fluctuating terms, one quadratic and the other linear, plus a fluctuating linear term proportional to z. The latter can make the minimum of the overall potential zero (extinction) or larger than zero (survival). Note that for fluctuating K_i, the LV potential is $V_i(N_i) = -\rho\left(K_i N_i - \frac{N_i^2}{2}\right)$ and the average in the 1replica computation above is performed over the Gaussian distributed K_i with mean $\mu_K = 1$ and variance σ_K.

Zero Temperature Limit

In the zero temperature limit $q_0 \to q_D$ at the same pace as T so it is useful to define the variable $\Delta q = \rho\beta(q_D - q_0)$, where ρ has been inserted just for convenience in writing. The equations of the three parameters in this limit are hence expressed in terms of h, q_0, and Δq.
In this limit the thermal averages over the 1replica measure are evaluated by saddle point-method at N^*, which is the positive minimum of the Hamiltonian $H_{\rm RS}(N)$ when it exists, or zero. Hence the SP equations become

$$q_0 = \overline{N^*(z)^2}\ , \qquad h = \overline{N^*(z)}\ , \qquad \Delta q = \rho\overline{\frac{\theta(N^*(z))}{H''_{\rm RS}(N^*(z))}}\ .$$

where $\theta(x)$ is the Heaviside function, $\theta(x) = 1$ for $x > 0$ and zero otherwise. Note that in the case of last equation on $\Delta q = \overline{\langle N^2\rangle - \langle N\rangle^2}$ we had to Taylor expand $H_{\rm RS}$ for small T separately in the case of extinction $N^* = 0$ and survival $N^* > 0$.
The LV case is particularly simple since N^* reads

$$N^*(z) = \max\left\{0, \frac{K + z\sigma\sqrt{q_0} - \mu h}{1 - \sigma^2\Delta q}\right\}\ . \tag{33}$$

Until now K and z were two separate Gaussian variables (with averages 1 and 0 and variances σ_K and 1, respectively) over which we are averaging. Combining together these two variables into $\widetilde{z}$ with 0 average and variance 1 we get

$$N^* = \max\left\{0, \frac{\sqrt{\sigma_K^2 + \sigma^2 q_0}}{1-\sigma^2\Delta q}(\widetilde{z}+\Delta)\right\} \tag{34}$$

with value of the random variable $\widetilde{z}$ corresponding to extinction $-\Delta = -\frac{1-\mu h}{\sqrt{q_0\sigma^2+\sigma_K^2}}$. The expression for q_0, h, and Δq are hence immediately obtained as being

$$q_0 = \left(\frac{\sqrt{q_0\sigma^2 + \sigma_K^2}}{1-\sigma^2\Delta q}\right)^2 w_2(\Delta)\ , \tag{35}$$

$$h = \frac{\sqrt{q_0\sigma^2 + \sigma_K^2}}{1-\sigma^2\Delta q} w_1(\Delta)\ , \tag{36}$$

and

$$\Delta q = \frac{1}{1-\sigma^2\Delta q} w_0(\Delta)\ , \tag{37}$$

with

$$w_i(\Delta) = \int_{-\Delta}^{\infty} \frac{d\widetilde{z}}{\sqrt{2\pi}} \exp\left[-\frac{\widetilde{z}^2}{2}\right](\widetilde{z}+\Delta)^i\ .$$

These equations coincide with the one obtained by the cavity method[3].

4 One step replica symmetry breaking equation

In the multiple minima phase the RS solution is unstable, as already checked beforehand[3]. This implies the existence of multiple equilibria. In order to characterize these equilibria one has to study what kind of RSB solution emerges. In the following we consider the 1RSB solution: the n replica are divided into n/m groups and $Q_{ab} = q_1$ for $a \neq b$ both in the same group, $Q_{ab} = q_0$ for a, b in different groups and $Q_{aa} = q_D$ and $H_a = h$. Once introduced this *ansatz* the computation is similar to the RS one.
The free-energy expression is in this case

$$-\beta n F = \ln \overline{\int dq_0 dq_1 dq_D dh \exp\left[S\mathcal{A}(q_0, q_1, q_D, h)\right]}^V \tag{38}$$

with

$$\begin{aligned}\mathcal{A}(q_0, q_1, q_D, h) = & -\rho^2\sigma^2\beta^2\frac{n}{4}[(n-m)q_0^2 + (m-1)q_1^2 + q_D^2] \\ & + \rho\mu\beta\frac{n}{2}h^2 + \frac{1}{S}\sum_i \ln Z_i\end{aligned} \tag{39}$$

with an effective Hamiltonian, $H_{eff}(\{N^a\}_i)$, for Z_i:

$$H_{eff} = \sum_a [\rho\mu h N_i^a + V_i(N_i^a) + (T-\lambda)\log N_i^a] - \frac{\beta\rho^2\sigma^2}{2} \times$$
$$\left[q_0\left(\sum_a^n N_i^a\right)^2 + \frac{n}{m}(q_1 - q_0)\left(\sum_a^m N_i^a\right)^2 + (q_D - q_1)\sum_a N_i^{a\,2}\right].$$

To decouple replicas we exploit standard properties of Gaussian integrals and we get

$$
\begin{aligned}
Z_i = \int \frac{dz_i}{\sqrt{2\pi}} \exp\left[-\frac{z_i^2}{2}\right] \prod_{a_B=1}^{n/m} \Bigg(\int \frac{dz_{B,i}^{a_B}}{\sqrt{2\pi}} \prod_{a(a_B)} dN_i^a \times \\
\exp\left[-\frac{z_{B,i}^{a_B\,2}}{2} - \beta \sum_{a(a_B)} H_{1\mathrm{RSB}}(N_i^a, z_i, z_{B,i}) \right] \Bigg)
\end{aligned} \tag{40}
$$

with $a(a_B) \in [(a_B-1)m+1, a_B m]$ and

$$
\begin{aligned}
H_{1\mathrm{RSB}}\,(\, N, z, z_B) = -\rho^2\sigma^2\beta(q_D - q_1)\frac{N^2}{2} + (T-\lambda)\log N \\
+V(N) + \left(\rho\mu h - z_B\rho\sqrt{q_1-q_0}\sigma - z\rho\sqrt{q_0}\sigma\right) N \ .
\end{aligned} \tag{41}
$$

By maximizing the action $\mathcal{A}$ at the exponent of (38) in the $n \to 0$ limit we get the following SP equations on the introduced parameters

$$
h = \overline{\frac{1}{S}\sum_i \langle N_i^{a(a_B)} \rangle_{\mathrm{AR}}}^V
$$

$$
q_0 = \overline{\frac{1}{S}\sum_i \langle N_i^{a(a_B)} N_i^{a(b_B)} \rangle_{\mathrm{AR}}}^V
$$

$$
q_1 = \overline{\frac{1}{S}\sum_i \langle N_i^{a(a_B)} N_i^{b(a_B)} \rangle_{\mathrm{AR}}}^V
$$

and

$$
q_D = \overline{\frac{1}{S}\sum_i \langle {N_i^{a(a_B)}}^2 \rangle_{\mathrm{AR}}}^V
$$

with

$$
\overline{\langle N^{b(b_B)^p} N^{c(b_B)^r} N^{d(d_B)^s} \rangle_{\mathrm{AR}}}^V =
$$

$$
= \frac{\overline{\int d\mu(z, z_B^{a_B}, N^{a(a_B)}) \exp\left[-\beta \sum_{a(a_B)} H_{1\mathrm{RSB}}^{a(a_B)}\right] N^{b(b_B)^p} N^{c(b_B)^r} N^{d(d_B)^s}}^V}{\int d\mu(z, z_B^{a_B}, N^{a(a_B)}) \exp\left[-\beta \sum_{a(a_B)} H_{1\mathrm{RSB}}^{a(a_B)}\right]}
$$

where

$$
d\mu = \frac{dz \exp\left[-\frac{z^2}{2}\right]}{\sqrt{2\pi}} \prod_{a_B} \frac{dz_B^{a_B} \exp\left[-\frac{z_B^{a_B\,2}}{2}\right]}{\sqrt{2\pi}} \prod_{a(a_B)} dN^{a(a_B)}
$$

and $H_{1\mathrm{RSB}}^{a(a_B)} = H_{1\mathrm{RSB}}(N^{a(a_B)}, z, z_B^{a_B})$.
These averages in the $n \to 0$ limit can be expressed in terms of thermal averages

$\langle\cdot\rangle_{\mathrm{1R}}$ over single species and single replica with Hamiltonian $H_{\mathrm{1RSB}}(N,z,z_B)$

$$\langle\cdot\rangle_{\mathrm{1R}} = \frac{\int dN \exp\left[-\beta H_{\mathrm{1RSB}}(N,z,z_B)\right]\ \cdot}{\int dN \exp\left[-\beta H_{\mathrm{1RSB}}(N,z,z_B)\right]}\ ,$$

averages $\langle\cdot\rangle_{\mathrm{mR}}$ over the Gaussian variable z_B with additional weight given by $\left(\int dN \exp\left[-\beta H_{\mathrm{1RSB}}(N,z,z_B)\right]\right)^m$

$$\langle\cdot\rangle_{\mathrm{mR}} = \frac{\int \frac{dz_B}{\sqrt{2\pi}} \exp\left[-\frac{{z_B}^2}{2}\right] \left(\int dN \exp\left[-\beta H_{\mathrm{1RSB}}(N,z,z_B)\right]\right)^m \cdot}{\int \frac{dz_B}{\sqrt{2\pi}} \exp\left[-\frac{{z_B}^2}{2}\right] \left(\int dN \exp\left[-\beta H_{\mathrm{1RSB}}(N,z,z_B)\right]\right)^m}$$

and averages $\overline{\cdot}^V$ representing the average over the disorder contained in $V(N)$ and the Gaussian integral over z with mean zero and unit variance. Using all this we have

$$\begin{aligned} &\overline{\langle N^{b(b_B)^p} N^{c(b_B)^r} N^{d(d_B)^s}\rangle_{\mathrm{AR}}}^V = \\ &\overline{\langle\langle N^p\rangle_{\mathrm{1R}}\langle N^r\rangle_{\mathrm{1R}}\rangle_{\mathrm{mR}}\langle\langle N^s\rangle_{\mathrm{1R}}\rangle_{\mathrm{mR}}}\ . \end{aligned} \tag{42}$$

Hence we can write

$$q_0 = \overline{\langle\langle N\rangle_{\mathrm{1R}}\rangle^2_{\mathrm{mR}}} \tag{43}$$

$$q_1 = \overline{\langle\langle N\rangle^2_{\mathrm{1R}}\rangle_{\mathrm{mR}}} \tag{44}$$

$$q_D = \overline{\langle\langle N^2\rangle_{\mathrm{1R}}\rangle_{\mathrm{mR}}} \tag{45}$$

$$h = \overline{\langle\langle N\rangle_{\mathrm{1R}}\rangle_{\mathrm{mR}}}\ . \tag{46}$$

1RSB Zero Temperature Limit

Also in the case we have to considered rescaled variable in the limit $T \to 0$: $\rho(q_D - q_1)\beta = \Delta q \sim O(1)$, and the scaling of the replica breaking order parameter m is such that βm remains of the order of one. Hence, in the following we will introduce the notation $\beta m = \widetilde{m}$ and keep $\widetilde{m} \sim O(1)$.
In this limit, similarly to the RS case, the SP equations read as follows:

$$q_0 = \overline{\langle N^*\rangle^2_{\mathrm{mR}}} \tag{47}$$

$$q_1 = \overline{\langle {N^*}^2\rangle_{\mathrm{mR}}} \tag{48}$$

$$\Delta q = \rho\overline{\langle\frac{\theta(N^*)}{H''_{\mathrm{1RSB}}(N^*)}\rangle_{\mathrm{mR}}} \tag{49}$$

$$h = \overline{\langle N^*\rangle_{\mathrm{mR}}}\ . \tag{50}$$

Previously though the expressions were even simpler because we had to compute averages of the kind

$$\int \frac{dz}{\sqrt{2\pi}} \exp[-z^2/2] \frac{\left(\int dN \exp\left[-\beta H_{\mathrm{RS}}(N,z)\right] N^l\right)^k}{\left(\int dN \exp\left[-\beta H_{\mathrm{RS}}(N,z)\right]\right)^k}$$

hence, thanks to the evaluation the integral on N by the saddle point, $\exp[-\beta H_{\mathrm{RS}}]$ in the numerator and denominator cancel out.

In this case instead an additional weight depending on m should be considered next to the Gaussian weight on z_B which comes from $\exp[-\beta m H_{1\mathrm{RSB}}(N^*(z,z_B),z,z_B)]$ when $N(z,z_B)^* \neq 0$. For the same reason the normalization constant is non trivial and must be evaluated.
The LV choice of V allows for simple explicit expression of the equations. In particular as in the LV RS case, we combine the Gaussian variables z and K into $\widetilde{z}$ and for every z_B we get

$$N^* = \max\left\{0, \frac{\sigma\sqrt{q_1-q_0}}{1-\sigma^2\Delta q}(z_B+\Delta(\widetilde{z}))\right\} \tag{51}$$

with the new value of the random variable z_B corresponding to extinction

$$-\Delta(\widetilde{z}) = -\frac{\widetilde{z}\sqrt{\sigma_K^2+\sigma^2 q_0}+1-\mu h}{\sigma\sqrt{q_1-q_0}} \ .$$

For every given z_B the additional weight involving $H_{1\mathrm{RSB}} = H_{1\mathrm{RSB}}(N^*,\widetilde{z},z_B)$ is

$$\exp[-\widetilde{m}H_{1\mathrm{RSB}}] = \exp\left[\frac{\widetilde{m}}{2}\frac{\rho\sigma^2(q_1-q_0)}{1-\sigma^2\Delta q}(z_B+\Delta(\widetilde{z}))^2\right]$$

when N^* is non null. Hence the normalization constant is

$$\int\frac{dz_B}{\sqrt{2\pi}}\exp[-z_B^2/2]\left(\int dN\exp[-\beta H_{1\mathrm{RSB}}(N,\widetilde{z},z_B)]\right)^m = A(\widetilde{z})+D(\widetilde{z})$$

with

$$A(\widetilde{z}) = \int_{-\Delta(\widetilde{z})}^{\infty}\frac{dz_B}{\sqrt{2\pi}}\exp\left[\frac{\widetilde{m}}{2}\frac{\rho\sigma^2(q_1-q_0)}{1-\sigma^2\Delta q}(z_B+\Delta(\widetilde{z}))^2-\frac{z_B^2}{2}\right]$$

and

$$D(\widetilde{z}) = \int_{-\infty}^{-\Delta(\widetilde{z})}\frac{dz_B}{\sqrt{2\pi}}\exp\left[-\frac{z_B^2}{2}\right] \ .$$

With this in mind and defining

$$d\mu(z_B;\widetilde{z}) = \frac{dz_B}{\sqrt{2\pi}}\exp\left[-\frac{z_B^2}{2}+\frac{\widetilde{m}}{2}\frac{\rho\sigma^2(q_1-q_0)}{1-\sigma^2\Delta q}(z_B+\Delta(\widetilde{z}))^2\right]$$

we can finally write the 1RSB self consistence equations as follows

$$h = \int\frac{d\widetilde{z}}{\sqrt{2\pi}}\exp\left[-\frac{\widetilde{z}^2}{2}\right]\frac{B(\widetilde{z})}{A(\widetilde{z})+D(\widetilde{z})}$$

with

$$B(\widetilde{z}) = \int_{-\Delta(\widetilde{z})}^{\infty}d\mu(z_B;\widetilde{z})\frac{\sigma\sqrt{q_1-q_0}}{1-\sigma^2\Delta q}(z_B+\Delta(\widetilde{z})) \ ,$$

$$q_0 = \int\frac{d\widetilde{z}}{\sqrt{2\pi}}\exp[-\widetilde{z}^2/2]\frac{B(\widetilde{z})^2}{(A(\widetilde{z})+D(\widetilde{z}))^2} \ ,$$

$$q_1 = \int\frac{d\widetilde{z}}{\sqrt{2\pi}}\exp[-\widetilde{z}^2/2]\frac{C(\widetilde{z})}{A(\widetilde{z})+D(\widetilde{z})}$$

with

$$C(\widetilde{z}) = \int_{-\Delta(\widetilde{z})}^{\infty} d\mu(z_B;\widetilde{z}) \frac{\sigma^2(q_1-q_0)}{(1-\sigma^2\Delta q)^2}(z_B+\Delta(\widetilde{z}))^2 \ ,$$

and

$$\begin{aligned}\Delta q &= \rho\beta\overline{\langle\langle N^2\rangle_{1\mathrm{R}} - \langle N\rangle^2_{1\mathrm{R}}\rangle_{mR}} \\ &= \frac{1}{1-\sigma^2\Delta q}\int \frac{d\widetilde{z}}{\sqrt{2\pi}} \exp\left[-\frac{\widetilde{z}^2}{2}\right]\frac{A(\widetilde{z})}{A(\widetilde{z})+D(\widetilde{z})} \ .\end{aligned} \tag{52}$$

Everywhere we could determine also the $\widetilde{m}$ given by a SP equation, which satisfies the following condition

$$\begin{aligned}0 = \widetilde{m}^2(q_1^2-q_0^2)\frac{\rho^2\sigma^2}{4} + \int \frac{dz}{\sqrt{2\pi}} \exp\left[-\frac{z^2}{2}\right] \times \\ \left[\log(A(z)+D(z)) - \frac{\rho\widetilde{m}(1-\sigma^2\Delta q)}{2}\frac{C(z)}{A(z)+D(z)}\right] \ .\end{aligned} \tag{53}$$

What we do is instead to use $\widetilde{m}$ as a parameter through which we can select minima of the 1RSB structure at different energy levels. This allows to compute the number of minima with a given energy, using $\widetilde{m}$ as a parameter conjugated to the energy[4]. The logarithm of the number of minima divided by S is called configurational entropy. It is proportional[4] to the derivative of the free energy with respect to m. Note that, by definition of $\widetilde{m}$, the configurational entropy of minima corresponding to the equilibrium in the 1RSB phase is null.
In the $n \to 0$ and $T \to 0$ limit the free-energy reads

$$\begin{aligned}-F &= \frac{1}{\beta n}\ln\overline{Z^n} \\ &= S\left[-\frac{\rho\sigma^2}{4}[\widetilde{m}\rho(q_1^2-q_0^2)+2q_1\Delta q] + \frac{\rho\mu}{2}h^2\right. \\ &\left.+\frac{1}{\widetilde{m}}\int\frac{dz}{\sqrt{2\pi}}\exp[-z^2/2]\log(A(z)+D(z))\right]\end{aligned} \tag{54}$$

and the configurational entropy is

$$\begin{aligned}S_c &= -m^2\frac{d}{dm}\left(\frac{1}{n}\ln\overline{Z^n}\right) \\ &= \widetilde{m}^2(q_1^2-q_0^2)\frac{\rho^2\sigma^2}{4} + \int\frac{dz}{\sqrt{2\pi}}\exp[-z^2/2] \times \\ &\left[\log(A(z)+D(z)) - \frac{\rho\widetilde{m}(1-\sigma^2\Delta q)}{2}\frac{C(z)}{A(z)+D(z)}\right] \ .\end{aligned} \tag{55}$$

5 Instability of the 1RSB phase and marginality condition for the FRSB phase

We now study the (in)stability of the 1RSB phase and, more generally, obtain the condition for the stability of RSB phases.
To obtain the stability condition we consider a generic $k-RSB$ phase and study

fluctuations δQ_{ab} only inside the inner blocks of the Parisi matrix. This is the so-called replicon eigenvalue and corresponds physically to fluctuations within a state. As we shall discuss in the next section, this is directly related to the Hessian (stability matrix) around one equilibrium. We call L the action that has to be extremized at the saddle-point and we study its Hessian with respect to the fluctuations described above:

$$\begin{aligned}\frac{\partial^2 L}{\partial Q_{ab}\partial Q_{cd}} =& (\beta\rho\sigma)^2 \delta_{\langle ab\rangle,\langle cd\rangle} \\ &- (\beta\rho\sigma)^4 \left(\overline{\langle N^a N^b N^c N^d\rangle} - \overline{\langle N^a N^b\rangle\langle N^c N^d\rangle}\right)\end{aligned} \tag{56}$$

where the average is done over the effective Hamiltonian. All replica indices belong to the same block of size $m \times m$, one of the inner ones. A single inner block is analogous to a replica $m \times m$ symmetric matrix so the corresponding Hessian matrix can be diagonalized rather easily (see Almeida and Thouless). Within the same block there are three independent matrix elements depending whether some replica index are the same:

$$\begin{aligned}P &= (\beta\rho\sigma)^2 - (\beta\rho\sigma)^4 \left(\overline{\langle (N^a)^2 (N^b)^2\rangle} - \overline{\langle N^a N^b\rangle^2}\right) \\ Q &= -(\beta\rho\sigma)^4 \left(\overline{\langle (N^a)^2 N^b N^c\rangle} - \overline{\langle N^a N^b\rangle^2}\right) \\ R &= -(\beta\rho\sigma)^4 \left(\overline{\langle N^a N^b N^c N^d\rangle} - \overline{\langle N^a N^b\rangle^2}\right) \ .\end{aligned}$$

The replicon eigenvalue[5] is $\lambda = P - 2Q + R$ with degeneracy $m(m-1)/3$. The condition for stability is $\lambda \geq 0$. Marginal stability corresponds to $\lambda = 0$.
To evaluate the replicon eigenvalue in the 1RSB phase we need to consider a single block a_B and evaluate

$$\begin{aligned}\overline{\langle (N^a)^2 (N^b)^2\rangle_{AR}}^V &= \overline{\langle\langle N^2\rangle_{1\mathrm{R}}^2\rangle_{\mathrm{mR}}} \\ \overline{\langle (N^a)^2 N^b N^c\rangle_{AR}}^V &= \overline{\langle\langle N^2\rangle_{1\mathrm{R}}\langle N\rangle_{1\mathrm{R}}^2\rangle_{\mathrm{mR}}} \\ \overline{\langle N^a N^b N^c N^d\rangle_{AR}}^V &= \overline{\langle\langle N\rangle_{1\mathrm{R}}^4\rangle_{\mathrm{mR}}} \ .\end{aligned}$$

Hence the replicon eigenvalue can be expressed in a more transparent way as

$$\lambda = (\beta\sigma\rho)^2 \left[1 - (\beta\sigma\rho)^2 \overline{\langle(\langle N^2\rangle_{1\mathrm{R}} - \langle N\rangle_{1\mathrm{R}}^2)^2\rangle_{\mathrm{mR}}}\right]$$

where the second moment of N^2 within one single state (or equilibrium) appears. Using the fluctuation dissipation relation one can rewrite the previous equation in term of single species responses:

$$\lambda = (\beta\sigma\rho)^2 \left[1 - (\sigma\rho)^2 \overline{\langle\left(\frac{\partial N}{\partial \xi}\right)^2\rangle}\right]$$

In the FRSB phase the replicon is exactly zero, this is related to the criticality of the phase[2], thus one obtain the equation:

$$(\sigma\rho)^2 \overline{\langle\left(\frac{\partial N}{\partial \xi}\right)^2\rangle} = 1$$

which encodes the marginality condition at finite temperature. In the small T limit this computation is analogous to the one performed for Δq. At the end one gets the simpler expression

$$(\sigma\rho)^2 \overline{\langle \theta(N^*) \left(\frac{1}{H''_{1\mathrm{RSB}}(N^*)} \right)^2 \rangle_{\mathrm{mR}}} = 1 \ .$$

or its equivalent expression in terms of single species response

$$(\sigma\rho)^2 \overline{\langle \theta(N^*) \left(\frac{\partial N}{\partial \xi} \right)^2 \rangle_{\mathrm{mR}}} = 1 \ .$$

which leads to Eq. (5) of the main text. For the usual Lotka-Volterra case in which $V(N)$ is quadratic one gets $H''_{1\mathrm{RSB}}(N^*) = \rho(1 - \sigma^2 \Delta q)$ which does not depend on N^*. Thus the replicon eigenvalue and consequently the marginality condition is particularly simple:

$$\overline{\langle \theta(N^*) \rangle_{\mathrm{mR}}} \frac{\sigma^2}{(1 - \sigma^2 \Delta q)^2} = 1$$

As spotted earlier, the expression of Δq in terms of correlation function is very similar. In the LV case the similarity becomes even closer:

$$\begin{aligned} \Delta q &= \rho\beta \overline{\langle \langle N^2 \rangle_{1\mathrm{R}} - \langle N \rangle_{1\mathrm{R}}^2 \rangle_{\mathrm{mR}}} \\ &= \rho \overline{\langle \theta(N^*) \frac{1}{H''_{1\mathrm{RSB}}(N^*)} \rangle_{\mathrm{mR}}} \\ &= \overline{\langle \theta(N^*) \rangle_{\mathrm{mR}}} \frac{1}{(1 - \sigma^2 \Delta q)} \ . \end{aligned} \tag{57}$$

Using together the equation on Δq and the marginality condition one finds two simpler appealing expressions:

$$\phi\sigma^2 = \frac{1}{4} \qquad \sigma^2 \Delta q = \frac{1}{2}$$

where $\phi = \overline{\langle \theta(N^*) \rangle}$. The first equation is the the limit of stability given by the May Bound: the fraction of surviving species in any equilibrium should be such that the Wigner semi-circle touches zero. The second is a general result valid in the marginal phase. These analytical predictions have been tested in Fig. 2 and Fig. 3 of the main text.

6 Phase diagram and numerical solution of the mean-field equations

The replica symmetric phase had been already studied[3]. Our results agree with the previous one. In particular we find three phases, see Supplementary Fig. 1. By increasing σ for $\mu > 0$ the single equilibrium phase becomes unstable toward the multiple equilibria (spin-glass) phase when its replicon eigenvalue vanishes. The instability toward the unbounded growth phase is signalled by a concomitant divergence of $\overline{\langle N \rangle}$ and $\overline{\langle N^2 \rangle - \langle N \rangle^2}$. Note that the transition line

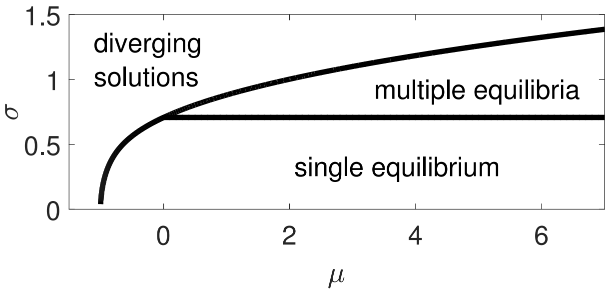

Figure 1: Phase Diagram obtained analytically from the mean-field equations.

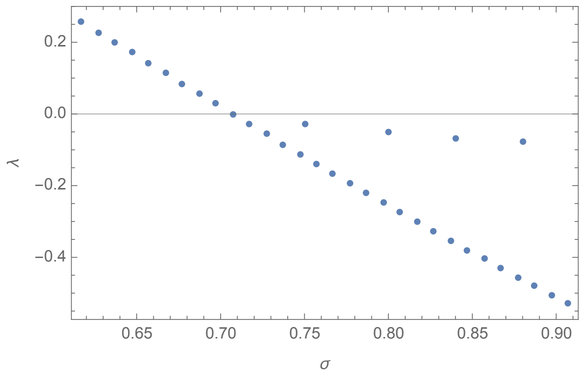

Figure 2: Replicon eigenvalue plotted for the RS and the 1RSB phases for $\mu = 2$ as a function of σ. The 1RSB eigenvalue corresponds to the topmost points for $\sigma > \sigma_c = 1/\sqrt{2} \simeq 0.707$.

to the unbounded growth phase was determined within the RS ansatz so it is only an approximation for $\mu > 0$. We have checked by numerical simulations that it is actually a good approximation.

Crossing the transition toward the multiple equilibria phase one finds that the RS phase becomes unstable and one has to break replica symmetry. We have found that also the 1RSB solution is unstable even though much less than the RS one, see Supplementary Fig. 2 where the replicon eigenvalue is plotted for the RS and the 1RSB phases for the standard LV model with $r_i = K_i = 1$ for $\mu = 2$ as a function of σ. We didn't look for 2RSB solutions and directly assumed that the stable phase if the FRSB one as found generically in spin-glass models[2]. We validated this assumption by comparison with numerical simulations that show marginal stability in the multiple equilibria phase, a property valid only for the FRSB phase.

Note that, although unstable, the 1RSB provides a very good approximation as we have checked by comparison with numerical simulations. For example, in Supplementary Fig. 3 and Supplementary Fig. 4 we show $\frac{\partial N_i^*}{\partial \xi_i^*} = \frac{1}{1-\sigma^2 \Delta q}$ and $\phi\sigma^2$ for the standard LV model with $r_i = K_i = 1$ for $\mu = 2$. These two quantities have respectively to stick to the values 2 and 1/4 as discussed in the main text and found by numerical simulations. As shown the 1RSB is already a very good approximation of the correct results, corresponding to the FRSB

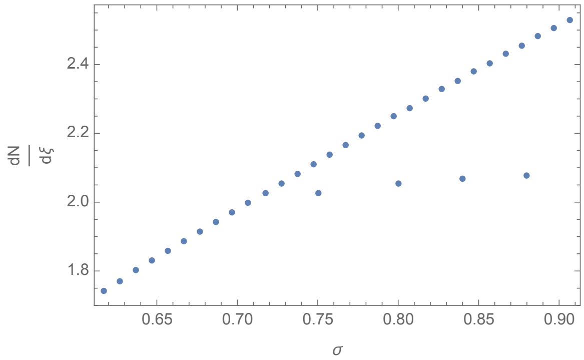

Figure 3: Single species response as a function of σ for the standard LV model with $r_i = K_i = 1$ for $\mu = 2$ from the RS and 1RSB solution. The 1RSB result corresponds to the bottom points for $\sigma > \sigma_c = 1/\sqrt{2} \simeq 0.707$. The correct FRSB result is $\frac{\partial N_i^*}{\partial \xi_i^*} = 2$ for $\sigma > \sigma_c$.

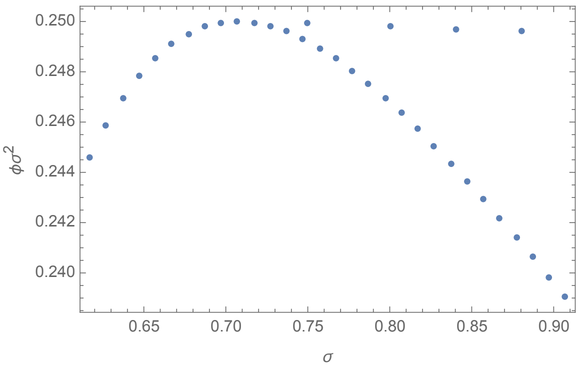

Figure 4: $\phi\sigma^2$ as a function of σ for the standard LV model with $r_i = K_i = 1$ for $\mu = 2$ from the RS and 1RSB solution. For $\sigma > \sigma_c$ both phases are unstable but the 1RSB result is very close to the correct one corresponding to $\phi\sigma^2 = 1/4$.

phase. We have also computed the configurational entropy. Given that the 1RSB is unstable, we cannot determine even approximatively the most numerous equilibria[2]. The values we found for the configurational entropy as a function of energy within the 1RSB ansatz for the standard LV model with $r_i = K_i = 1$ for $\mu = 2$ and $\sigma = 0.88$ are very small, in the range $10^{-3} - 10^{-4}$. It would be interesting (but also quite involved) to obtain the correct result within a FRSB computation. Anyhow, it is important to keep in mind that the number of equilibria in realistic situations can be modest depending on the value of the configurational entropy and the total number of species.

7 Random Matrix Analysis

As explained in the text, the stability of a given equilibrium is governed by the $S^* \times S^*$ stability matrix M_{ij}^* which is defined by the equation $(M^*)^{-1}_{ij} = \frac{\partial N_i^*}{\partial \xi_j^*}$

and reads

$$M^*_{ij} = V''(N^*_i)\delta_{ij} + \alpha_{ij} \tag{58}$$

In order to study its spectral properties we focus on the resolvent, defined as $G(\lambda) = (S^*)^{-1}\mathrm{Tr}(\lambda\mathbf{1} - \mathrm{M}^*)^{-1}$ and add an infinitesimal negative imaginary part to λ. Following exactly the same procedure developed for mean-field spin-glasses in[6], one can construct a perturbative expansion in α_{ij}, sum the leading contributions for $S \to \infty$ and obtain the equation:

$$\left(\frac{1}{\lambda\mathbf{1} - M^*}\right)_{ii} = \frac{1}{\lambda - V''(N^*_i) - \sigma^2\phi G(\lambda)} \tag{59}$$

valid only for indices corresponding to surviving species. By summing over the surviving species one finds

$$G(\lambda) = \left\langle \frac{1}{\lambda - V''(N^*_i) - \sigma^2\phi G(\lambda)} \right\rangle \tag{60}$$

where the average is over the distribution of the $V''(N^*_i)s$. This equation allows one to study the density of eigenvalues $\rho(\lambda)$ of M^* thanks to the relation $\mathrm{ImG}(\lambda) = \pi\rho(\lambda)$. Since eqs. (59,60) are also the equations satisfied by the resolvent of a random matrix $\tilde{M}^*$ with independent identically distributed Gaussian off-diagonal entries having the same first and second moment of α_{ij}, and independent identically distributed diagonal entries with the same statistics of $V''(N^*_i)$, we conclude that M^* and $\tilde{M}^*$ are equivalent as far as the average spectrum is concerned (a relation that we checked explicitly by numerics). A marginally stable equilibrium is characterized by arbitrary small eigenvalues of its stability matrix, i.e. it is such the left edge of the support of $\rho(\lambda)$ is zero. This implies that $\mathrm{ImG}(\lambda)$ becomes arbitrary small for $\lambda \to 0$. In consequence, close to $\lambda = 0$, we can develop the self-consistent equation on the resolvent as:

$$\begin{aligned}\mathrm{ImG}(\lambda) &= \sigma^2\phi \left\langle \left(\frac{1}{\lambda - \mathrm{V}''(\mathrm{N}^*_\mathrm{i}) - \sigma^2\phi\mathrm{RG}(\lambda)}\right)^2 \right\rangle \mathrm{ImG}(\lambda) + \\ &-(\sigma^2\phi)^3 \left\langle \left(\frac{1}{\lambda - V''(N^*_i) - \sigma^2\phi\mathrm{RG}(\lambda)}\right)^4 \right\rangle (\mathrm{ImG}(\lambda))^3 + \cdots\end{aligned} \tag{61}$$

Given the type of random matrix we are focusing on, we do not expect any isolated eigenvalue popping out of the spectrum. Therefore, the condition for marginal stability can be obtained from the bulk density of eigenvalues. The condition for having a non-zero imaginary part is that when collecting all terms on the RHS the coefficient on the linear term in ImG is positive for $\lambda > 0$ and vanishes at $\lambda = 0$. This leads to the equation

$$1 = \sigma^2\phi \left\langle \left(\frac{1}{V''(N^*_i) + \sigma^2\phi\mathrm{RG}(\lambda)}\right)^2 \right\rangle$$

Using relation (59), and replacing $\left(\frac{1}{V''(N^*_i)+\sigma^2\phi\mathrm{RG}(\lambda)}\right)$ by $(M^*)^{-1}_{ii}$ in the identity above, we obtain the equation for marginal stability quoted in the text:

$$\phi\sigma^2 \left(\frac{1}{S^*}\sum_{i=1}^{S^*} \left((M^*)^{-1}_{ii}\right)^2\right) = 1$$

8 Dynamical four-point correlation function $\chi_4(t,t')$

In the following we derive the analytical results quoted in the main text on $\chi_4(t,t')$ in the limit of small noise. First, we define the $S^* \times S^*$ matrix A as

$$A \equiv (\alpha^*)^{-1}$$

Let's call also N_i^* the abundance of the surviving species in the limit of zero noise. For small noise their abundances have fluctuations of the order $\sqrt{T}$ around the zero-noise value, whereas instead abundances of species with $N_i^* = 0$ have fluctuations of the order T. For small noise and at leading order, one can do a quadratic approximation of the Hamiltonian and find for the surviving species:

$$\langle \delta N_i(t)\, \delta N_j(t) \rangle = T A_{ij}$$

where

$$\delta N_i(t) = N_i(t) - N_i^* \ .$$

The definitions of $C(t,t')$ and χ_4 read

$$C(t,t') = \left\langle \frac{1}{S} \sum_i \delta N_i(t)\, \delta N_i(t') \right\rangle \tag{62}$$

$$\chi_4(t,t') = \frac{S}{C(t,t)^2} \left\langle \left(\frac{1}{S} \sum_i \delta N_i(t)\, \delta N_i(t') \right)^2 \right\rangle - S \left[\frac{C(t,t')}{C(t,t)} \right]^2 \ . \tag{63}$$

The correlation at equal time is related (for small noise) to α^* via

$$C(t,t) = \left\langle \frac{1}{S} \sum_i \delta N_i(t)\, \delta N_i(t') \right\rangle = \frac{T}{S} Tr\left[A\right] \tag{64}$$

Note that the species characterized by zero abundance in the zero noise limit do not contribute at leading order in T since they would give a contribution $O(T^2)$. At long times the correlation function vanishes

$$C(t \to \infty, t') = 0$$

This also happens for the correlation between different species:

$$\langle \delta N_i(t)\, \delta N_j(t') \rangle = 0 \ .$$

so

$$\lim_{t\to\infty} \chi_4(t,t') = \lim_{t\to\infty} \left(\frac{1}{\frac{T}{S} Tr\left[A\right]} \right)^2 \left\langle S \left(\frac{1}{S} \sum_i \delta N_i(t)\, \delta N_i(t') \right)^2 \right\rangle$$

$$= \lim_{t\to\infty} \left(\frac{S}{T\ Tr\left[A\right]} \right)^2 \frac{1}{S} \sum_{i,j} \langle \delta N_i(t)\, \delta N_j(t) \rangle \langle \delta N_i(t')\, \delta N_j(t') \rangle$$

$$= \frac{S}{\left(Tr\left[A\right]\right)^2} \sum_{i,j=1}^{S^*} A_{ij}^2 = \frac{S}{\left(Tr\left[A\right]\right)^2} \sum_i \left[A^2\right]_{ii} = \frac{S Tr\left[A^2\right]}{\left(Tr\left[A\right]\right)^2}$$

while for $t = t'$ repeating an analogous computation one finds

$$\chi_4\left(t,t\right) = \frac{2STr\left[A^2\right]}{\left(Tr\left[A\right]\right)^2} = 2\chi_4\left(t\to\infty,t'\right)$$

The trace can be related to the spectrum via

$$\frac{1}{S^*}Tr\left[A^n\right] = \frac{1}{S^*}Tr\left[\left(\alpha^*\right)^{-n}\right] = \int d\lambda \frac{\rho\left(\lambda\right)}{\lambda^n}$$

For a semi-circle of radius a centered at b,

$$\rho\left(\lambda\right) = \frac{2}{\pi a}\sqrt{1-\frac{(\lambda-b)^2}{a^2}}\ ,$$

this gives

$$\frac{1}{S}Tr\left[\left(\alpha^*\right)^{-2}\right] = \phi\int d\lambda\frac{\rho\left(\lambda\right)}{\lambda^2} = \frac{2\phi}{a^2}\left(\frac{b}{\sqrt{b^2-a^2}}-1\right)\ .$$

And for the correlations

$$\frac{1}{S}Tr\left[\left(\alpha^*\right)^{-1}\right] = \phi\int d\lambda\frac{\rho\left(\lambda\right)}{\lambda} = \frac{2\phi}{a^2}\left(b-\sqrt{b^2-a^2}\right)$$

In the standard LV parameterization, the center is at $b = 1$ and $a = 2\sigma\sqrt{\phi}$, so one finds

$$\frac{C\left(t,t\right)}{T} = \frac{1}{S}Tr\left[A\right] = \frac{1}{2\sigma^2}\left(1-\sqrt{1-4\phi\sigma^2}\right)$$

and

$$\chi_4\left(t,t\right) = 2\frac{\frac{1}{\sqrt{1-4\phi\sigma^2}}-1}{1-\sqrt{1-4\phi\sigma^2}}\qquad,\qquad \chi_4\left(t\to\infty,t'\right) = \frac{1}{2}\chi_4\left(t,t\right)$$

As discussed in the main text and shown in Fig. 5, $C(t,t)$ is featureless at the transition while $\chi_4(t,t)$ diverges.

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