SOME ASYMPTOTIC FORMULAE IN THE THEORY OF PARTITIONS (II)

By G. SZEKERES (*Adelaide*)

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1. Introduction

LET $P(n,k)$ denote the number of partitions of n into at most k integer parts, or what is the same, into parts not exceeding k. In an earlier paper† I have determined the asymptotic behaviour of $P(n,k)$ when k is not too large, viz. $0{\cdot}135k^2 \leqslant n$. In the present paper I shall obtain an asymptotic formula which is valid for arbitrarily large values of k. The main result to be proved is

THEOREM 1. *Let n/k^2 be bounded, $n \leqslant c_1 k^2$, and let β, v be determined from*

$$v\beta = k, \qquad \beta^2 \int\limits_0^v \frac{t}{e^t-1}\,dt + \tfrac{1}{2}\beta\Big(\frac{v}{e^v-1}-1\Big) + \frac{1}{12}\Big(\frac{1}{2}+\frac{1}{e^v-1}-\frac{ve^v}{(e^v-1)^2}\Big) = n. \tag{1}$$

Then, uniformly in n and k,

$$P(n,k) = \frac{1}{2\pi} B_0^{-\frac{1}{2}}\beta^{-2}\exp\Big\{2\beta\int\limits_0^v \frac{t}{e^t-1}\,dt - (v\beta+\tfrac{1}{2})\log(1-e^{-v}) + \frac{1}{2}\Big(\frac{v}{e^v-1}-1\Big)\Big\}[1+B_1(v)\beta^{-1}+\ldots+B_{m-1}(v)\beta^{-m+1}+O(\beta^{-m})] \tag{2}$$

for any given $m > 0$, where

$$B_0 = \int\limits_0^v \frac{t^2e^t}{(e^t-1)^2}\,dt = 2\int\limits_0^v \frac{t}{e^t-1}\,dt - \frac{v^2}{e^v-1} \tag{3}$$

and $B_\mu(v) = O(1)$.

Remarks:

(i) Throughout the paper, $c_1, c_2,\ldots, C_1, C_2,\ldots$ denote positive constants, independent of k and n but possibly depending on some other previously fixed constants such as the bound for n/k^2, or m in the expansion (2). The same remark applies to constants appearing in the O-notations, e.g. $B_\mu(v) = O(1)$ means $|B_\mu(v)| \leqslant C_\mu$. The letter c without any suffix is reserved for the constant $c = 6^{\frac{1}{2}}/\pi$.

† *Quart. J. of Math.* (Oxford) (2) 2 (1951), 85–108, subsequently quoted as PI.

Quart. J. Math. Oxford (2), 4 (1953), 96–111.

(ii) $v \geqslant c_2$. For, equation (1) implies

$$v^{-2}\int\limits_0^v \frac{t}{e^t-1}\,dt = n/k^2+\tfrac{1}{2}k^{-1}\Big(\frac{1}{v}-\frac{1}{e^v-1}\Big)-\tfrac{1}{12}k^{-2}\Big(\frac{1}{2}+\frac{1}{e^v-1}-\frac{ve^v}{(e^v-1)^2}\Big)$$
$$\leqslant c_1+(4kv)^{-1} \tag{4}$$

since $$n \leqslant c_1 k^2, \qquad \frac{1}{v}-\frac{1}{e^v-1} \leqslant \frac{1}{2}$$

and the last term is negative for $v > 0$. But the left-hand side of (4) is certainly greater than $1/2v$ when $0 < v < 1$. Hence (4) implies

$$1 < 2c_1 v+(2k)^{-1} < 2c_1 v+\tfrac{1}{2}, \qquad v > (4c_1)^{-1}.$$

(iii) $c_3 n^{\frac{1}{2}} \leqslant \beta \leqslant c_4 n^{\frac{1}{2}}$. This follows from the fact that the coefficients of β^2, β, and β^0 on the left-hand side of (1) are bounded, i.e. $O(1)$ for $v \geqslant 0$, and

$$c_5 \leqslant \int\limits_0^v \frac{t}{e^t-1}\,dt \leqslant c^{-2}, \text{ by remark (ii).}$$

This shows that the expansion (2) proceeds essentially by powers of $n^{-\frac{1}{2}}$.

(iv) It would be possible to determine the functions $B_\mu(v)$ if desired, but it seems difficult to get a general explicit formula. For $B_1(v)$ one obtains

$$B_1(v) = -\big[\tfrac{1}{12}ve^v(e^v-1)^{-2}+B_0^{-1}\{\tfrac{1}{8}+\tfrac{1}{4}v^2e^v(e^v-1)^{-2}\}+$$
$$+B_0^{-2}\{\tfrac{1}{8}v^4(e^v+e^{2v})(e^v-1)^{-3}-\tfrac{3}{4}v^3e^v(e^v-1)^{-2}\}+$$
$$+\tfrac{5}{24}B_0^{-3}v^6e^{2v}(e^v-1)^{-4}\big]. \tag{5}$$

The expression on the right is bounded since $B_0 \geqslant c_6$ for $v \geqslant c_2$ by (3).

(v) Let $v = c^{-1}\lambda n^{\frac{1}{2}}$ ($\lambda \geqslant \delta$), where δ is a fixed positive number. Then

$$\int\limits_0^v \frac{t}{e^t-1}\,dt = c^{-2}-\int\limits_v^\infty te^{-t}[1+O\{\exp(-c^{-1}\delta n^{\frac{1}{2}})\}]\,dt$$
$$= c^{-2}+O\{n^{\frac{1}{2}}\exp(-c^{-1}\delta n^{\frac{1}{2}})\},$$

and one obtains from (1), (3), and (5)

$$\beta = c(n-\tfrac{1}{24})^{\frac{1}{2}}+\tfrac{1}{4}c^2+\tfrac{1}{32}c^3n^{-\frac{1}{2}}+O(n^{-1}),$$
$$\beta^{-1} = c^{-1}(n-\tfrac{1}{24})^{-\frac{1}{2}}\{1-\tfrac{1}{4}cn^{-\frac{1}{2}}+O(n^{-1})\},$$
$$B_0^{-\frac{1}{2}} = 2^{-\frac{1}{2}}c+O\{n\exp(-c^{-1}\delta n^{\frac{1}{2}})\},$$
$$B_1 = -\tfrac{1}{16}c^2+O\{n^2\exp(-c^{-1}\delta n^{\frac{1}{2}})\}.$$

Hence

$$P(n,k) = \{2^{\frac{3}{2}}\pi c(n-\tfrac{1}{24})\}^{-1}\exp\big(2c^{-1}(n-\tfrac{1}{24})^{\frac{1}{2}}\big)\{1-\tfrac{1}{2}cn^{-\frac{1}{2}}+O(n^{-1})\}$$

for $k = v\beta = \lambda n+\frac{1}{4}c\lambda n^{\frac{1}{2}}+O(1) \geqslant 2\delta n$. This is the well-known asymptotic term of the Hardy–Ramanujan formula.

The proof of Theorem 1 will be given in the next section. It is entirely independent of the theory of elliptic modular functions and also of Sylvester's theory of 'waves' which formed the core of the proof of my earlier formula in PI. In fact, it depends only on Cauchy's theorem and Euler's summation formula and is essentially an application of the so-called 'method of steepest descent'.

If n/k^2 is not bounded, i.e. $k = o(n^{\frac{1}{2}})$, then formula (2) must be slightly modified. The asymptotic term is still correct, provided that $k \to \infty$, but the error terms have a different form. Formula (29) of § 3 shows that the first error term is $O(k^{-1})$†.

In § 4 I shall investigate $p(n, k)$, the number of partitions of n into exactly k positive integer parts, and $q(n, k)$, the number of partitions of n into k unequal parts. The following is the main result:

THEOREM 2. *If n is sufficiently large, then there exists a number $k_1 = k_1(n)$ such that $p(n, k) < p(n, k+1)$ for $k < k_1$, $p(n, k) > p(n, k+1)$ for $k > k_1$. The value of k_1 is*

$$k_1 = cn^{\frac{1}{2}}L + c^2(\tfrac{3}{2} + \tfrac{3}{2}L - \tfrac{1}{4}L^2) - \tfrac{1}{2} + O(n^{-\frac{1}{2}}\log^4 n), \tag{6}$$

where $L = \log(cn^{\frac{1}{2}})$, $c = 6^{\frac{1}{2}}/\pi = 0{\cdot}7796968...$, $c^2 \doteq 0{\cdot}6079271...$.

THEOREM 3. *If n is sufficiently large, then there exists a number $k_0 = k_0(n)$ such that $q(n, k) < q(n, k+1)$ for $k < k_0$, $q(n, k) > q(n, k+1)$ for $k > k_0$. The value of k_0 is*

$$k_0 = (2^{\frac{1}{2}}\log 2)cn^{\frac{1}{2}} + 2b(\log 2)^{-1} - \frac{\frac{1}{2}b}{1-2b} - 1 + O(n^{-\frac{1}{2}}), \tag{7}$$

where $b \doteq c^2(\log 2)^2$.

Numerically, $k_0 = 0{\cdot}7643041n^{\frac{1}{2}} - 0{\cdot}5084280 + O(n^{-\frac{1}{2}})$.

Theorem 2 proves a conjecture of Auluck, Chowla, and Gupta‡, and improves a formula of Erdős§ who showed that

$$k_1 = cn^{\frac{1}{2}}\log(cn^{\frac{1}{2}}) + o(n^{\frac{1}{2}}).$$

It also disproves a conjecture I made in PI, 94.

Theorem 3 was essentially proved in PI. My earlier k_0 was greater by $\frac{1}{2}$ than the present value, owing to a different interpretation of the maxima.

† The formula in PI had a different form and proceeded according to powers of k^{-1}.

‡ *J. Indian Math. Soc.* 6 (1942), 105–12.

§ *Bull. American Math. Soc.* 52 (1946), 185–8. Erdős uses the notation $c = \pi(\frac{2}{3})^{\frac{1}{2}}$.

2. Proof of Theorem 1

Let
$$F(w) = \prod_{\nu=1}^{k}(1-w^\nu)^{-1} = \sum_{n=0}^{\infty} P(n,k)w^n. \tag{8}$$

By Cauchy's theorem
$$P(n,k) = (2\pi i)^{-1}\int F(w)w^{-n-1}\,dw \tag{9}$$

taken along a circle with centre at O and radius $\rho < 1$. I choose ρ to be the real positive root of the equation $wF'/F(w) = n$, i.e.
$$\sum_{\nu=1}^{k}\frac{\nu\rho^\nu}{1-\rho^\nu} = \sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1} = n, \qquad \alpha = -\log\rho. \tag{10}$$

The essential point is that with this choice of ρ we shall be able to evaluate the integral (9) quite accurately in the neighbourhood of the 'saddle point' $w = \rho$ and then show that the more remote parts of the path do not contribute substantially to the value of the integral. This can be done quite easily, without having recourse to Farey dissections or similar devices. Thus the influence of the secondary singularities is automatically eliminated.

Throughout this section the symbol $A(u)$ denotes some unspecified continuous function of u, bounded for $u \geqslant 0$, which, if desired, can be determined explicitly. In the following I shall frequently use Euler's summation formula†

$$f(1)+...+f(k) = \int_0^k f(x)\,dx+\tfrac{1}{2}\{f(k)-f(0)\}+$$
$$+\sum_{\nu=1}^{m}\frac{1}{(\nu+1)!}B_{\nu+1}\{f^{(\nu)}(k)-f^{(\nu)}(0)\}+\int_0^k P_{m+1}(x)f^{(m+1)}(x)\,dx,$$

where $B_2 = \frac{1}{6}$, $B_3 = 0,...,$ generally B_ν is the νth Bernoulli number, and $P_m(t)$ is the mth Bernoulli polynomial.

Take
$$f(x) = x(e^{\alpha x}-1)^{-1} = \frac{1}{\alpha}t(e^t-1)^{-1} = \frac{1}{\alpha}\phi(t),$$

where $t = \alpha x$, so that
$$f^{(\nu)}(x) = \alpha^{\nu-1}\phi^{(\nu)}(t) \quad\text{and}\quad f^{(\nu)}(0) = \alpha^{\nu-1}B_\nu.$$

† See K. Knopp, *Infinite Series* (London 1928), 526.

Clearly $\phi^{(\nu)}(t)$ is bounded for $t \geqslant 0$ and tends to zero monotonically for $t \geqslant t_\nu$. Hence, for a fixed m,

$$\int_0^\infty |\phi^{(m+1)}(t)|\,dt = O(1)$$

and

$$\int_0^k |P_{m+1}(x)f^{(m+1)}(x)|\,dx = \alpha^{m-1}\int_0^{\alpha k}\left|P_{m+1}\left(\frac{1}{\alpha}t\right)\phi^{(m+1)}(t)\right|dt = O(\alpha^{m-1}).$$

On writing $u = \alpha k$, Euler's formula and equation (10) give

$$n = \sum_{\nu=1}^{k}\nu(e^{\nu\alpha}-1)^{-1} = \alpha^{-2}\int_0^u \frac{t}{e^t-1}\,dt+\tfrac{1}{2}\alpha^{-1}\left(\frac{u}{e^u-1}-1\right)+$$

$$+\frac{1}{12}\left(\frac{1}{e^u-1}-\frac{ue^u}{(e^u-1)^2}+\frac{1}{2}\right)+\alpha^2A(u)+\ldots+\alpha^{m-2}A(u)+O(\alpha^{m-1}). \quad (11)$$

Generally, for $0 \leqslant s < r$, one obtains

$$\sum_{\nu=1}^{k}\nu^r e^{s\alpha\nu}(e^{\alpha\nu}-1)^{-r} = \alpha^{-r-1}\int_0^u t^r e^{st}(e^t-1)^{-r}\,dt+$$

$$+\alpha^{-r}A(u)+\ldots+\alpha^{-r+m-1}A(u)+O(\alpha^{-r+m}). \quad (12)$$

Comparing (1) and (11) we find

$$\left.\begin{aligned} \alpha &= \beta^{-1}\{1+A(v)\beta^{-4}+\ldots+A(v)\beta^{-m+1}+O(\beta^{-m})\} \\ u &= v\{1+A(v)\beta^{-4}+\ldots+A(v)\beta^{-m+1}+O(\beta^{-m})\} \end{aligned}\right\}. \quad (13)$$

The second relation is a consequence of the first one since $u = k\alpha = \alpha\beta v$. To prove the first relation, put $\alpha = \beta^{-1}(1+\epsilon)$, $u = v(1+\epsilon)$ into (11) and subtract equation (1). The result is

$$0 = -2\beta^2\epsilon\int_0^v \frac{t}{e^t-1}\,dt+\beta^2\epsilon\frac{v^2}{e^v-1}+\beta^2O(\epsilon^2)+\beta O(\epsilon)+\beta^{-2}A(v)$$

$$= -\beta^2\epsilon\int_0^v t^2e^t(e^t-1)^{-2}\,dt+\beta^{-2}A(v)+\beta^2O(\epsilon^2)+\beta O(\epsilon),$$

which implies $\epsilon = \beta^{-4}A(v)+O(\beta^{-5})$. This proves (13) for $m = 4$. In the general case we use induction on m. Writing

$$\epsilon = \beta^{-4}A(v)+\ldots+\beta^{-m+1}A(v)+\epsilon_1, \qquad \epsilon_1 = O(\beta^{-m})$$

one obtains as before

$$-\beta^2\epsilon_1 \int_0^v t^2e^t(e^t-1)^{-2}\,dt+\beta^{-m+2}A(v)+O(\beta^{-m+1}) = 0,$$

where $A(v)$ can be determined recursively. It follows that ϵ_1 has the form $\beta^{-m}A(v)+O(\beta^{-m-1})$.

From (8) and (9) we get

$$P(n,k) = (2\pi)^{-1}\rho^{-n}F(\rho) \int_{-\pi}^{\pi} \frac{F(\rho e^{i\theta})}{F(\rho)}\, e^{-ni\theta}\,d\theta$$

$$= (2\pi)^{-1}\exp(n\alpha+\log F(\rho)) \int_{-\pi}^{\pi} \exp\left(-\sum_{\nu=1}^{k}\log\frac{1-\rho^\nu e^{i\nu\theta}}{1-\rho^\nu}-ni\theta\right)d\theta. \quad (14)$$

The integral is dissected in three parts:

$$\int_{-\pi}^{\pi} = I_1+I_2+I_3, \qquad I_1 = \int_{-\theta_0}^{\theta_0}, \qquad I_2 = \int_{\theta_0}^{\pi}, \qquad I_3 = \int_{-\pi}^{-\theta_0}, \quad (15)$$

where $\theta_0 = n^{-5/7}$.

For the integrand in I_1 we can write

$$\exp\left[-\left\{\sum_{\nu=1}^{k'}+\sum_{\nu=k'+1}^{k}\right\}\log\frac{1-\rho^\nu e^{i\nu\theta}}{1-\rho^\nu}-ni\theta\right], \quad (16)$$

where $k' = \min\{k,[n^{3/4}]\}$.† The second sum in (16) is empty unless $k > n^{3/4}$, when $(k'+1)\alpha > c_7 n^{1/4} > 1$ for $n > c_8$ and

$$\left|\sum_{\nu=k'+1}^{k}\right| \leqslant \sum_{\nu=k'+1}^{\infty}\log\{1+2(e^{\alpha\nu}-1)^{-1}\} < 2\sum_{\nu=k'+1}^{\infty}(e^{\alpha\nu}-1)^{-1}$$

$$< 3\sum_{\nu=k'+1}^{\infty}e^{-\alpha\nu} = O(e^{-\alpha(k'+1)})$$

$$= O\{\exp(-c_9 n^{1/4})\}. \quad (17)$$

† $[x]$ denotes the greatest integer not exceeding x.

Since $|\nu\theta| \leqslant k'\theta_0 \leqslant n^{-1/21}$, the first sum in (16) can be expanded in powers of $i\nu\theta$,

$$\sum_{\nu=1}^{k'} \log\frac{1-\rho^\nu e^{i\nu\theta}}{1-\rho^\nu}$$

$$= -\sum_{\nu=1}^{k'} \log\Big\{1-(e^{\nu\alpha}-1)^{-1}\sum_{\mu=1}^{m}\frac{1}{\mu!}(i\nu\theta)^\mu + O[(\nu|\theta|)^{m+1}(e^{\nu\alpha}-1)^{-1}]\Big\}$$

$$= i\theta\sum_{\nu=1}^{k'}\frac{\nu}{e^{\nu\alpha}-1} - \tfrac{1}{2}\theta^2\sum_{\nu=1}^{k'}\frac{\nu^2 e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2} + \sum_{\mu=3}^{2m+1}\frac{1}{\mu!}(i\theta)^\mu\sum_{\nu=1}^{k'}\frac{\nu^\mu g_\mu(e^{\nu\alpha})}{(e^{\nu\alpha}-1)^\mu} +$$

$$+O\Big\{\theta^{2m+2}\sum_{\nu=1}^{\infty}\nu^{2m+2}\sum_{\mu=1}^{2m+2}(e^{\nu\alpha}-1)^{-\mu}\Big\}, \quad (18)$$

where $g_3(t) = t+t^2$, $g_4(t) = t+4t^2+t^3$, and generally $g_\mu(t)$ is a polynomial of degree $\mu-1$. Replacing the upper summation index k' by k, we see that the error is

$$O\Bigg(\int_{n^{1/3}}^{\infty}\frac{dt}{e^{t\alpha}}\Bigg) = O\{n^{\frac{1}{2}}\exp(-c_{10}\,n^{\frac{1}{6}})\},$$

and the first sum on the right of (18) becomes $ni\theta$.† The other sums can be replaced by the corresponding integrals, with the help of Euler's formula [equation (12)]. The result is

$$ni\theta - \tfrac{1}{2}\theta^2\alpha^{-3}\int_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt - \theta^2\alpha^{-3}\sum_{\nu=1}^{n-1}A(u)\alpha^\nu +$$

$$+\sum_{\mu=3}^{2m+1}(i\theta)^\mu\alpha^{-\mu-1}\sum_{\nu=0}^{m-1}A(u)\alpha^\nu + O(\theta^2\alpha^{m-3}+\theta^{2m+2}\alpha^{-2m-3}).$$

This and (15), (16), (17) give

$$I_1 = \int_{-\theta_0}^{\theta_0}\exp\Bigg(-\tfrac{1}{2}\theta^2\alpha^{-3}\int_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt\Bigg)\times$$

$$\times\Big\{1+\sum_{\mu=1}^{m}\theta^{2\mu}\alpha^{-[8\mu/3]}\sum_{\nu=0}^{m-1}A(u)\alpha^\nu + ih(\theta,\alpha) + O\Big(\sum_{\nu=1}^{m-1}\theta^{2\nu}\alpha^{m-3\nu}\Big)\Big\}\,d\theta,$$

where $h(\theta,\alpha)$ is an odd function of θ. Hence, writing

$$A_0 = \int_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt, \qquad x = A_0^{\frac{1}{2}}(2\alpha^3)^{-1}\theta,$$

† This is the decisive step where the saddle-point property of ρ is used.

we get

$$I_1 = (2\alpha^3 A_0^{-1})^{\frac{1}{2}} \int_{-x_0}^{x_0} e^{-x^2}\Big\{1 + \sum_{\mu=1}^{n} \Big(\frac{2}{A_0}\Big)^{\mu} \alpha^{[(\mu+2)/3]} x^{2\mu} \sum_{\nu=0}^{m-1} A(u)\alpha^{\nu} + \\ + O\Big(\alpha^{m} \sum_{\nu=1}^{m+1} x^{2\nu}\Big)\Big\}\, dx,$$

where $x_0 = A_0^{\frac{1}{2}}(2\alpha^3)^{-\frac{1}{2}}\theta_0 \geqslant c_{11}\, n^{3/4-5/7} = c_{11}\, n^{1/28}$.

Replacing the limits $\pm x_0$ by $\pm\infty$, we see that the error is at most

$$O\Big\{\int_{x_0}^{\infty} x^{2m+2} e^{-x^2}\, dx\Big\} = O\{n \exp(-c_{12}\, n^{1/14})\}.$$

Thus $\quad I_1 = (2\pi\alpha^3 A_0^{-1})^{\frac{1}{2}}\{1 + A_1(u)\alpha + \ldots + A_{m-1}(u)\alpha^{m-1} + O(\alpha^m)\} \qquad (19)$

for certain functions $A_\mu(u)$ which are bounded for $u \geqslant c_{13}$. In fact, each $A_\mu(u)$ can be expressed as a fraction $a_\mu(u)\{A_0(u)\}^{-3\mu}$, where $a_\mu(u)$ is bounded for $u \geqslant 0$ and $A_0(u) \geqslant c_{14}$ for $u \geqslant c_{13}$. A simple calculation gives, e.g.

$$-A_1(u) = A_0^{-1}\Big(\frac{1}{8} + \frac{1}{4}\,\frac{u^2 e^{u}}{(e^{u}-1)^2}\Big) + A_0^{-2}\Big(\frac{1}{8}\,\frac{u^4(e^{u}+e^{2u})}{(e^{u}-1)^3} - \frac{3}{4}\,\frac{u^3 e^{u}}{(e^{u}-1)^2}\Big) + \\ + \frac{5}{24} A_0^{-3}\, \frac{u^6 e^{2u}}{(e^{u}-1)^4}. \qquad (20)$$

It remains to be shown that I_2 and I_3 in (15) are negligible. For the absolute value $G(\theta)$ of the integrand in I_2 one has, if $n^{-5/7} \leqslant \theta \leqslant n^{-1/2}$ and $k' = \min(k, [\alpha^{-1}], [\pi/4\theta])$,

$$G(\theta) = \Big|\exp\Big(\sum_{\nu=1}^{k} \log \frac{1-\rho^{\nu}}{1-\rho^{\nu} e^{i\nu\theta}}\Big)\Big|$$

$$\leqslant \exp\Big[-\sum_{\nu=1}^{k'} \tfrac{1}{2} \log\{1 + 2e^{\nu\alpha}(e^{\nu\alpha}-1)^{-2}(1-\cos\nu\theta)\}\Big]$$

$$\leqslant \exp\Big[-\tfrac{1}{2}\sum_{\nu=1}^{k'} \log\{1+\nu^2\theta^2/(8\nu^2\alpha^2)\}\Big] = \exp[-\tfrac{1}{2}k' \log\{1+\theta^2/(8\alpha^2)\}],$$

$$G(\theta) < \exp(-c_{15}\, n^{1/14}), \qquad (21)$$

since

$$\tfrac{1}{2}k' \log(1+\theta^2/8\alpha^2) > \tfrac{1}{32}k'\theta^2\alpha^{-2} \geqslant \tfrac{1}{32}[\alpha]^{-3}\theta_0^2 > c_{16}\, n^{3/2-10/7} \\ = c_{16}\, n^{1/14} \quad \text{for } \theta \leqslant 8\alpha$$

and again

$$\tfrac{1}{2}k' \log(1+\theta^2/8\alpha^2) > \tfrac{1}{2}k' \log(\theta/4\alpha) > c_{17}\, n^{\frac{1}{2}} \quad \text{for } 8\alpha < \theta \leqslant n^{-\frac{1}{2}}.$$

If $n^{-\frac{1}{2}} < \theta \leqslant \pi$, then for every positive integer $m \leqslant m_0 = k'\theta/2\pi$, where $k' = \min(k, [\alpha^{-1}])$, there is at least one $\nu \leqslant k'$ with

$$(2m-\tfrac{3}{2})\pi \leqslant \nu\theta \leqslant (2m-\tfrac{1}{2})\pi,$$

and, for this ν,

$$\begin{aligned}\log|(1-\rho^{\nu})(1-\rho^{\nu}e^{i\nu\theta})^{-1}| &\leqslant \tfrac{1}{2}\log(1-\rho^{\nu})^2(1+\rho^{2\nu})^{-1} \\ &< \tfrac{1}{2}\log(e^{2\pi\alpha m}-1)^2(e^{4\pi\alpha m}+1)^{-1}.\end{aligned}$$

Hence

$$\begin{aligned}\log G(\theta) &\leqslant \tfrac{1}{2}\sum_{m\leqslant m_0}\log(e^{2\pi\alpha m}-1)^2(e^{4\pi\alpha m}+1)^{-1} \\ &\leqslant \tfrac{1}{2}\int_1^{m_0}\log(e^{2\pi\alpha x}-1)^2(e^{4\pi\alpha x}+1)^{-1}\,dx \\ &= (4\pi^{\alpha})^{-1}\int_{2\pi\alpha}^{2\pi\alpha m_0}\log\frac{(e^t-1)^2}{e^{2t}+1}\,dt < -c_{18}\,n^{\frac{1}{2}}\end{aligned}$$

since $2\pi\alpha m_0 = \alpha k'\theta \geqslant c_{19}\,n^{-\frac{1}{2}}$ and $2\pi\alpha \leqslant c_{20}\,n^{-\frac{1}{2}}$.

It follows that

$$G(\theta) < \exp(-c_{18}\,n^{\frac{1}{2}}) \quad \text{for } n^{-\frac{1}{2}} < \theta \leqslant \pi. \tag{22}$$

Now (21) and (22) give

$$G(\theta) < \exp(-c_{21}\,n^{1/14}) \quad \text{for } \theta_0 \leqslant \theta \leqslant \pi$$

and

$$|I_2| = |I_3| < 2\pi\exp(-c_{21}\,n^{1/14}). \tag{23}$$

Summarizing, (8), (10), (14), (15), (19), (23) give

$$\begin{aligned}P(n,k) = (2\pi A_0)^{-\frac{1}{2}}\alpha^{\frac{1}{2}}\exp\Bigl[\sum_{\nu=1}^{k}\Bigl\{\frac{\nu\alpha}{e^{\nu\alpha}-1}-\log(1-e^{-\nu\alpha})\Bigr\}\Bigr]\times \\ \times\{1+A_1(u)\alpha+\ldots+A_{m-1}(u)\alpha^{m-1}+O(\alpha^m)\},\end{aligned} \tag{24}$$

where $A_1(u)$ is given by (20).

To evaluate the expression in the exponential we once more use Euler's formula

$$\begin{aligned}\sum_{\nu=1}^{k}\log\frac{\nu}{1-e^{-\nu\alpha}} &= \int_0^k\log\frac{x}{1-e^{-\alpha x}}\,dx+\tfrac{1}{2}\{\log k-\log(1-e^{-u})+\log\alpha\}+ \\ &\quad+\tfrac{1}{12}\alpha\Bigl(\frac{1}{u}-\frac{1}{e^u-1}-\frac{1}{2}\Bigr)+\sum_{\mu=3}^{m-1}A(u)\alpha^{\mu}+O(\alpha^m) \\ &= (k+\tfrac{1}{2})\log k-k+\alpha^{-1}\int_0^u\frac{t}{e^t-1}\,dt-(k+\tfrac{1}{2})\log(1-e^{-u})+ \\ &\quad+\tfrac{1}{2}\log\alpha+\tfrac{1}{12}\alpha\Bigl(\frac{1}{u}-\frac{1}{e^u-1}-\frac{1}{2}\Bigr)+\sum_{\mu=3}^{m-1}A(u)\alpha^{\mu}+O(\alpha^m).\end{aligned}$$

This with Stirling's formula† gives

$$-\sum_{\nu=1}^{k}\log(1-e^{-\nu\alpha}) = \sum_{\nu=1}^{k}\log\frac{\nu}{1-e^{-\nu\alpha}}-\log k!$$

$$= \frac{1}{\alpha}\int_{0}^{u}\frac{t}{e^{t}-1}\,dt-\left(\frac{u}{\alpha}+\frac{1}{2}\right)\log(1-e^{-u})+\tfrac{1}{2}\log\frac{\alpha}{2\pi}-$$

$$-\tfrac{1}{12}\alpha\left(\frac{1}{2}+\frac{1}{e^{u}-1}\right)+\sum_{\mu=3}^{m-1}A(u)\alpha^{\mu}+O(\alpha^{m}).$$

Formula (2) with (4) is obtained from this, (11), and (24), by using (13) and (20).

3. The case $k = O(n^{\frac{1}{2}})$

We now drop the assumption that n/k^{2} is bounded and suppose that $k \leqslant c_{22}n^{\frac{1}{2}}$ but $k \to \infty$. Equation (11) gives $u \leqslant c_{23}$ and

$$n = k^{2}u^{-2}\int_{0}^{u}\frac{t}{e^{t}-1}\,dt+\tfrac{1}{2}k\left(\frac{1}{e^{u}-1}-\frac{1}{u}\right)+$$

$$+\frac{1}{12}\left(\frac{1}{e^{u}-1}+\frac{1}{2}-\frac{ue^{u}}{(e^{u}-1)^{2}}\right)+O(u^{2}k^{-2})$$

$$= k^{2}u^{-1}B_{1}(u)+kB_{2}(u)+\tfrac{1}{12}uB_{3}(u)+O(u^{2}k^{-2}), \tag{25}$$

where B_1, B_2, B_3 are bounded for $u \geqslant 0$, and $B_1(u) \geqslant c_{24}$ for $0 \leqslant u \leqslant c_{23}$. Therefore

$$u = O(k^{2}n^{-1}),\quad u^{-1} = O(nk^{-2}),\quad \alpha = u/k = O(k/n),\quad \alpha^{-1} = O(n/k), \tag{26}$$

and the error term in (25) is $O(k^{2}n^{-2})$.

If v, β are determined from

$$v = \beta k,\qquad n = k^{2}v^{-2}B_{1}(v)+kB_{2}(v)+\tfrac{1}{12}vB_{3}(v), \tag{27}$$

then

$$u = v\{1+O(k^{2}n^{-3})\} = v\{1+O(n^{-2})\},\qquad \alpha = \beta^{-1}\{1+O(n^{-2})\}. \tag{28}$$

† See K. Knopp, *Infinite Series*, 531.

To carry out the dissection (15) we take $\theta_0 = n^{-1}k^{4/7}$. Then $k\theta_0 = O(k^{-3/7})$ and (16) becomes

$$\exp\left\{-\sum_{\nu=1}^{k}\log\frac{1-\rho^\nu e^{i\nu\theta}}{1-\rho^\nu}-ni\theta\right\}$$

$$=\exp\left[-\sum_{\nu=1}^{k}\log\left\{1-\frac{1}{e^{\nu\alpha}-1}(i\nu\theta-\tfrac{1}{2}\nu^2\theta^2-\tfrac{1}{6}i\nu^3\theta^3)+O\left(\frac{\nu^4\theta^4}{e^{\nu\alpha}-1}\right)\right\}-ni\theta\right]$$

$$=\exp\left[-\tfrac{1}{2}\theta^2\sum_{\nu=1}^{k}\frac{\nu^2e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2}-\tfrac{1}{6}i\theta^3\sum_{\nu=1}^{k}\frac{\nu^3(e^{2\nu\alpha}+e^{\nu\alpha})}{(e^{\nu\alpha}-1)^3}+\right.$$

$$\left.+O\left\{\theta^4\sum_{\nu=1}^{\infty}\nu^4\sum_{\mu=1}^{4}(e^{\nu\alpha}-1)^{-\mu}\right\}\right]$$

$$=\exp\left\{-\tfrac{1}{2}\theta^2\alpha^{-3}\int\limits_0^u\frac{t^2e^t}{(e^t-1)^2}\,dt\right\}\left\{1-\tfrac{1}{6}i\theta^3\alpha^{-4}\int\limits_0^u\frac{t^3(e^{2t}+e^t)}{(e^t-1)^3}\,dt+\right.$$

$$\left.+O(n\theta^2+n^4k^{-3}\theta^4+n^6k^{-4}\theta^6)\right\}.$$

This gives, since $x_0=(\tfrac{1}{2}A_0)^{\frac{1}{2}}\alpha^{-\frac{3}{2}}\theta_0\geqslant c_{24}\,k^{1/14}$,

$$I_1=(2\pi\alpha^3A_0^{-1})^{\frac{1}{2}}\{1+O(k^{-1})\}.$$

To show that I_2 is small, we can use the same estimates as for (21) and (22) except that the intervals for θ are now $\theta_0\leqslant\theta\leqslant k^{-\frac{3}{2}}$ and $k^{-\frac{3}{2}}<\theta\leqslant\pi$ respectively. Hence

$$P(n,k)=(2\pi A_0)^{-\frac{1}{2}}\alpha^{\frac{3}{2}}\exp\left\{\sum_{\nu=1}^{k}\left(\frac{\nu\alpha}{e^{\nu\alpha}-1}\right)-\log(1-e^{-\nu\alpha})\right\}\{1+O(k^{-1})\}$$

$$=\frac{v^2}{2\pi k^2}B_0^{-\frac{1}{2}}\exp\left\{\frac{2k}{v}\int\limits_0^v\frac{t}{e^t-1}\,dt+\frac{1}{2}\left(\frac{v}{e^v-1}-1\right)-\right.$$

$$\left.-(k+\tfrac{1}{2})\log(1-e^{-v})\right\}\{1+O(k^{-1})\}, \quad (29)$$

where $B_0=\int\limits_0^v\frac{t^2e^t}{(e^t-1)^2}\,dt$ and v is determined from (27). This is valid uniformly in k provided that $k\to\infty$ and $k\leqslant c_{22}\,n^{\frac{1}{2}}$.

4. The functions $p(n, k)$ and $q(n, k)$

Formulae (2) and (29) respectively can be used as asymptotic expressions for $p(n, k)$ and $q(n, k)$, in virtue of the elementary relations

$$p(n,k) = P(n-k,k), \qquad q(n,k) = P\left\{n-\binom{k+1}{2}, k\right\}. \tag{30}$$

All we have to do is to replace the right-hand side of (1) by $n-k$ and $n-\binom{k+1}{2}$ respectively.

Consider first $p(n,k)$ and suppose that n/k^2 is bounded. Without loss of generality we may assume that $k < \frac{1}{2}n$ since $p(n,k) = P(n-k,k)$ for $k \geqslant \frac{1}{2}n$ by (30). (Hence $p(n,k)$ is steadily decreasing for $k \geqslant \frac{1}{2}n$ when n is fixed.) The last assumption implies that remark (iii) of the introduction is valid.

To obtain the position of the maximum of $p(n,k)$ when n is fixed, it is convenient to go back to the relations

$$\sum_{\nu=1}^{k} \frac{\nu}{e^{\nu\alpha}-1} = n-k, \tag{31}$$

$$p(n,k) = (2\pi A_0)^{-\frac{1}{2}}\alpha^{\frac{3}{2}}\exp\left\{\sum_{\nu=1}^{k} \frac{\nu\alpha}{e^{\nu\alpha}-1} - \log(1-e^{-\nu\alpha})\right\} \times$$
$$\times \{1+A_1(u)\alpha + A_2(u)\alpha^2 + O(\alpha^3)\}, \tag{32}$$

which follow from the equations (10) and (24).

Let $$\sum_{\nu=1}^{k+1} \frac{\nu}{e^{\nu(\alpha+\Delta\alpha)}-1} = n-k-1.$$

By (31) and Taylor's theorem,

$$\sum_{\nu=1}^{k} \frac{\nu}{e^{\nu\alpha}-1} - \sum_{\nu=1}^{k+1} \frac{\nu}{e^{\nu(\alpha+\Delta\alpha)}-1} = \sum_{\nu=1}^{k} \frac{\nu^2 e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2}\Delta\alpha - \frac{k+1}{e^{(k+1)\alpha}-1} +$$
$$+O\{u^2e^u\Delta\alpha + \alpha^{-4}(\Delta\alpha)^2\} = 1, \tag{33}$$

$$A_0\alpha^{-3}\Delta\alpha - \alpha^{-1}\frac{u}{e^u-1} = 1+O(ue^{-u}+\alpha^{-2}\Delta\alpha),$$

$$\Delta\alpha = A_0^{-1}\alpha^2\left(\frac{u}{e^u-1}+\alpha\right)+O(ue^{-u}\alpha^3+\alpha^4). \tag{34}$$

Also $u = \alpha k$, $u+\Delta u = (\alpha+\Delta\alpha)(k+1)$,

$$\Delta u = k\,\Delta\alpha+\alpha+\Delta\alpha = \frac{u}{\alpha}\Delta\alpha+\alpha+\Delta\alpha. \tag{35}$$

Furthermore,

$$\begin{aligned}\Delta \sum_{\nu=1}^{k} &\left\{\frac{\nu\alpha}{e^{\nu\alpha}-1}-\log(1-e^{-\nu\alpha})\right\}\\ &= -\alpha-\log(1-e^{-u})-\frac{\alpha}{e^{u}-1}-\tfrac{1}{2}A_0\,\alpha^{-3}(\Delta\alpha)^2+O(\Delta\alpha)\\ &= -\log(1-e^{-u})-\alpha\left\{1+\frac{1}{e^{u}-1}+\tfrac{1}{2}A_0^{-1}\frac{u^2}{(e^{u}-1)^2}\right\}+O(\Delta\alpha) \qquad (36)\end{aligned}$$

since

$$\begin{aligned}\Delta&\left\{\sum_{\nu=1}^{k}\frac{\nu\alpha}{e^{\nu\alpha}-1}\right\}\\ &= -\alpha+\sum_{\nu=1}^{k+1}\frac{\nu}{e^{\nu(\alpha+\Delta\alpha)}-1}\Delta\alpha\\ &= -\alpha+\left(\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1}\right)\Delta\alpha-\sum_{\nu=1}^{k}\frac{\nu^2e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2}(\Delta\alpha)^2+\\ &\qquad\qquad+\frac{k+1}{e^{(k+1)(\alpha+\Delta\alpha)}-1}\Delta\alpha+O\{\alpha^{-4}(\Delta\alpha)^3\}\\ &= -\alpha+\left(\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1}\right)\Delta\alpha-\frac{k}{e^{u}-1}\Delta\alpha-A_0\,\alpha^{-3}(\Delta\alpha)^2+O(\Delta\alpha),\end{aligned}$$

by (33) and (34), and

$$\begin{aligned}\Delta&\left\{-\sum_{\nu=1}^{k}\log(1-e^{-\nu\alpha})\right\}\\ &= -\log(1-e^{-(k+1)(\alpha+\Delta\alpha)})-\left(\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1}\right)\Delta\alpha+\\ &\qquad\qquad+\frac{1}{2}\sum_{\nu=1}^{k}\frac{\nu^2e^{\nu\alpha}}{(e^{\nu\alpha}-1)^2}(\Delta\alpha)^2+O\{\alpha^{-4}(\Delta\alpha)^3\}\\ &= -\left(\sum_{\nu=1}^{k}\frac{\nu}{e^{\nu\alpha}-1}\right)\Delta\alpha-\log(1-e^{-u})-\frac{\alpha}{e^{u}-1}-\frac{k}{e^{u}-1}\Delta\alpha+\\ &\qquad\qquad+\tfrac{1}{2}A_0\,\alpha^{-3}(\Delta\alpha)^2+O(\Delta\alpha).\end{aligned}$$

Finally

$$\tfrac{3}{2}\{\log(\alpha+\Delta\alpha)-\log\alpha\} = \tfrac{3}{2}\alpha^{-1}\Delta\alpha+O\{\alpha^{-2}(\Delta\alpha)^2\}$$

$$= \tfrac{3}{2}A_0^{-1}\alpha\frac{u}{e^u-1}+\tfrac{3}{2}A_0^{-1}\alpha^2+O(\Delta\alpha), \qquad (37)$$

$$-\tfrac{1}{2}\{\log A_0(u+\Delta u)-\log A_0(u)\}$$

$$= -\tfrac{1}{2}A_0^{-1}\frac{u^2e^u}{(e^u-1)^2}\left(\frac{u}{\alpha}\Delta\alpha+\alpha\right)+O\{(\Delta u)^2\}$$

$$= -\frac{\alpha}{2}\left\{A_0^{-1}\frac{u^2e^u}{(e^u-1)^2}+A_0^{-2}\frac{u^4e^u}{(e^u-1)^3}\right\}+O(u^2\Delta\alpha), \quad \text{by (35)}, \qquad (38)$$

and

$$\Delta\{\alpha A_1(u)+\alpha^2A_2(u)\} = O(u^2\Delta\alpha) \qquad (39)$$

by (20), (34), and (35). Hence

$$\Delta\log p(n,k) = \log p(n,k+1)-\log p(n,k)$$

$$= -\log(1-e^{-u})-\alpha\left[1+\frac{1}{e^u-1}+\tfrac{1}{2}A_0^{-1}\frac{u^2(e^u+1)}{(e^u-1)^2}-\tfrac{3}{2}A_0^{-1}\frac{u}{e^u-1}+\right.$$

$$\left.+\tfrac{1}{2}A_0^{-2}\frac{u^4e^u}{(e^u-1)^3}\right]+\tfrac{3}{2}A_0^{-1}\alpha^2+O(u^2\Delta\alpha) \qquad (40)$$

by (32), (36), (37), (38), and (39).

Put $u = \frac{1}{2}\log n+\lambda$, where λ is some function of n which either tends to $+\infty$ or $-\infty$ or remains bounded.

Since $c_{25}n^{-\frac{1}{2}} \leqslant \alpha \leqslant c_{26}n^{-\frac{1}{2}}$, the right-hand side of (40) is positive for sufficiently large n if $e^u = o(n^{\frac{1}{2}})$, i.e. if $\lambda \to -\infty$, and is negative if $e^{-u} = o(n^{-\frac{1}{2}})$, i.e. if $\lambda \to +\infty$. Hence we may assume that λ is bounded. Using

$$\alpha^{-2}\int\limits_0^u \frac{t}{e^t-1}\,dt+\tfrac{1}{2}\alpha^{-1}\left(\frac{u}{e^u-1}-1\right)+\frac{1}{12}\left(\frac{1}{2}+\frac{1}{e^u-1}-\frac{ue^u}{(e^u-1)^2}\right)$$

$$= n-\frac{u}{\alpha}+O(\alpha^2) \qquad (41)$$

and

$$\int\limits_0^u \frac{t}{e^t-1}\,dt = c^{-2}-\int\limits_u^\infty \frac{t}{e^t-1}\,dt = c^{-2}+u\log(1-e^{-u})-\sum_{k=1}^\infty k^{-2}e^{-ku}$$

$$= c^{-2}-(u+1)e^{-u}+O(ue^{-2u}), \qquad (42)$$

one obtains

$$\alpha = c^{-1}n^{-\frac{1}{2}}-\tfrac{1}{2}(ce^{-\lambda}-1)un^{-1}-\tfrac{1}{2}(\tfrac{1}{2}+ce^{-\lambda})n^{-1}+O(n^{-\frac{3}{2}}\log^2 n).$$

Noting that $A_0 = 2c^{-2}+O(n^{-\frac{1}{2}})$, the right-hand side of (40) becomes

$$(e^{-\lambda}-c^{-1})n^{-\frac{1}{2}}-\tfrac{1}{4}ce^{-\lambda}u^2n^{-1}+(\tfrac{5}{4}ce^{-\lambda}-\tfrac{1}{2})un^{-1}+$$
$$+(1+\tfrac{1}{2}ce^{-\lambda}-c^{-1}e^{-\lambda}+\tfrac{1}{2}e^{-2\lambda})n^{-1}+O(n^{-\frac{3}{2}}\log^4 n).$$

This becomes zero only if $ce^{-\lambda} = 1+\mu$, $\mu = O(n^{-\frac{1}{2}}\log^2 n)$,

$$u = \tfrac{1}{2}\log n+\lambda = L-\mu+O(n^{-1}\log^4 n),$$

where $L = \log(cn^{\frac{1}{2}})$, i.e.

$$\mu c^{-1}n^{-\frac{1}{2}}-\tfrac{1}{4}L^2n^{-1}+\tfrac{3}{4}Ln^{-1}+(\tfrac{3}{8}-\tfrac{1}{2}c^{-2})n^{-1} = O(n^{-\frac{3}{2}}\log^4 n),$$
$$\mu = \tfrac{1}{4}cL^2n^{-\frac{1}{2}}-\tfrac{3}{4}cLn^{-\frac{1}{2}}-(\tfrac{1}{2}c^{-1}-\tfrac{3}{8}c)n^{-\frac{1}{2}}+O(n^{-1}\log^4 n),$$
$$\alpha = c^{-1}n^{-\frac{1}{2}}-\tfrac{3}{4}n^{-1}+O(n^{-\frac{3}{2}}\log^3 n),$$
$$k = k_1 = u\alpha^{-1} = cn^{\frac{1}{2}}L-c^2(\tfrac{1}{4}L^2-\tfrac{3}{4}L-\tfrac{3}{8})-\tfrac{1}{2}+O(n^{-\frac{1}{2}}\log^4 n).$$

If k is less than k_1, the right-hand expression in (40) is positive; if k is greater than k_1, the expression is negative. This proves Theorem 2 for $n \leqslant c_1 k^2$.

For $k = o(n^{\frac{1}{2}})$ one has to use (29) instead of (32). One gets

$$\Delta \log p(n, k) = -\log(1-e^{-u})+O(k^{-1}),$$

which is positive since $u = O(1)$.

The proof of Theorem 3 is very similar except that (31) and (41) must be replaced by

$$\sum_{\nu=1}^{k} \frac{\nu}{e^{\nu\alpha-1}} = n-\binom{k+1}{2}, \qquad \sum_{\nu=1}^{k+1} \frac{\nu}{e^{\nu(\alpha+\Delta\alpha)}-1} = n-\binom{k+2}{2}, \tag{43}$$

and

$$\alpha^{-2}\int_0^u \frac{t}{e^t-1}\,dt+\tfrac{1}{2}\alpha^{-1}\left(\frac{u}{e^u-1}-1\right)+\tfrac{1}{2}u^2\alpha^{-2}+\tfrac{1}{2}u\alpha^{-1} = n+O(1). \tag{44}$$

From (43), $$\Delta\alpha = A_0^{-1}\alpha^2\frac{ue^u}{e^u-1}+O(\alpha^3),$$

$$\Delta\left[\sum_{\nu=1}^{k}\left\{\frac{\nu\alpha}{e^{\nu\alpha}-1}-\log(1-e^{-\nu\alpha})\right\}\right]$$
$$= -\log(e^u-1)-\alpha\left\{\frac{e^u}{e^u-1}+\tfrac{1}{2}A_0^{-1}\frac{u^2e^{2u}}{(e^u-1)^2}\right\}+O(\alpha^2),$$

$$\tfrac{3}{2}\Delta\{\log\alpha\} = \tfrac{3}{2}A_0^{-1}\alpha\frac{ue^u}{e^u-1}+O(\alpha^2),$$

$$-\tfrac{1}{2}\Delta\{\log A_0\} = -\tfrac{1}{2}\alpha\left\{A_0^{-1}\frac{u^2e^u}{(e^u-1)^2}+A_0^{-2}\frac{u^4e^{2u}}{(e^u-1)^3}+O(\alpha^2),\right.$$

$$\Delta\{\alpha A_1(u)\} = O(\alpha^2).$$

Hence

$$\Delta \log q(n,k) = -\log(e^{u}-1) - \alpha\left\{\frac{e^{u}}{e^{u}-1} + \tfrac{1}{2}A_0^{-1}\frac{u^2(e^{u}+e^{2u})}{(e^{u}-1)^2} - \right.$$

$$\left. - \tfrac{3}{2}A_0^{-1}\frac{ue^{u}}{e^{u}-1} + \tfrac{1}{2}A_0^{-2}\frac{u^4e^{2u}}{(e^{u}-1)^3}\right\} + O(\alpha^2),$$

which is zero only if $u = \log 2 + \lambda n^{-\frac{1}{2}}$ (λ bounded).

Writing $b = c^2\log^2 2$, we see from (44), in view of†

$$\int\limits_0^{\log 2} \frac{t}{e^t-1}\,dt = \tfrac{1}{2}c^{-2}(1-b) \quad \text{and} \quad A_0 = c^{-2}(1-2b) + O(n^{-\frac{1}{2}}),$$

that $\qquad \alpha = c^{-1}(2n)^{-\frac{1}{2}} + (\tfrac{1}{2}\log 2 - \tfrac{1}{4} + \lambda c 2^{\frac{1}{2}}\log 2)n^{-1} + O(n^{-\frac{3}{2}}).$

Hence

$$\Delta \log q(n,k) = \left\{-2\lambda - 2^{-\frac{1}{2}}c^{-1}\left(2 + 3\,\frac{b}{1-2b} - \right.\right.$$

$$\left.\left. - \frac{3b}{\log 2\,(1-2b)} + \frac{2b^2}{(1-2b)^2}\right)\right\} n^{-\frac{1}{2}} + O(n^{-1}),$$

which is zero only if

$$\lambda = -2^{-\frac{1}{2}}c^{-1}\left[1 + \frac{3}{2}\,\frac{b}{1-2b}\{1-(\log 2)^{-1}\} + \frac{b^2}{(1-2b)^2}\right] + O(n^{-\frac{1}{2}}),$$

$$k = k_0 = u\alpha^{-1} = 2^{\frac{1}{2}}(c\log 2)n^{\frac{1}{2}} - c^2\log 2(\log 2 - \tfrac{1}{2}) + 2^{\frac{1}{2}}c(1-2b)\lambda + O(n^{-\frac{1}{2}})$$

$$= 2^{\frac{1}{2}}(c\log 2)n^{\frac{1}{2}} - b + \tfrac{1}{2}b(\log 2)^{-1} - 1 +$$

$$+ 2b - \tfrac{3}{2}b\{1-(\log 2)^{-1}\} - \frac{b^2}{1-2b} + O(n^{-\frac{1}{2}})$$

$$= 2^{\frac{1}{2}}(c\log 2)n^{\frac{1}{2}} - 1 + 2b(\log 2)^{-1} - \frac{\frac{1}{2}b}{1-2b} + O(n^{-\frac{1}{2}}).$$

† D. Bierens de Haan, *Nouvelles tables d'intégrales définies* (New York, 1939), 151, formula (104, 5).