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Limit Theorems for the Number of Summands in Integer Partitions

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Central and local limit theorems are derived for the number of distinct summands in integer partitions, with or without repetitions, under a general scheme essentially due to Meinardus. The local limit theorems are of the form of Cramér-type large deviations and are proved by Mellin transform and the two-dimensional saddle-point method. Applications of these results include partitions into positive integers, into powers of integers, into integers $[j^{\beta}]$, $\beta > 1$, into $aj+b$, etc. © 2001 Academic Press

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1. INTRODUCTION

Let $\Lambda = \{\lambda_1, \lambda_2, ...\}$ be a sequence of positive integers satisfying $1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ and $\lambda_n \to +\infty$. There is an extensive literature on the asymptotics of the number of partitions of n into parts λ_j (see Andrews [1] and the references therein). In contrast, considerably fewer results have appeared in the literature on the limiting distribution of the number of summands (or parts) in random *restricted* (no part being repeated) or *unrestricted partitions* (repetition allowed). This paper is concerned with this aspect of the theory of partitions.

Erdős and Lehner [7] were the first to give a systematic study along this line in the case $\Lambda = \mathbb{Z}^+$. They showed that the number of summands (counted with multiplicities) in a random unrestricted partition of n follows asymptotically (as $n \to \infty$) an extreme-value distribution, a local version being later derived by Auluck *et al.* in [2]. Haselgrove and Temperley [15] extended, by a powerful analytic method, the result of Auluck *et al.* to more general Λ-partitions, their conditions on the given sequence being further extended, in some respects, only recently by Richmond [31]. Weaker results (convergence in distribution) under different analytic settings were derived by Lee [23] by the method of moments. A detailed

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study on the moments can be found in Richmond [28, 30]. It should be noted that the limiting distributions in these problems are all non-Gaussian. For many other extensions of the original problems, see for example [8, 9, 16, 24] and the references therein.

When each part is allowed to appear at most once, Erdős and Lehner [7] derived the asymptotic normality of the number of summands in the case $\Lambda = \mathbb{Z}^+$ (see also [9, 36, 37]). No extension of this result has appeared in the literature. In this paper, we consider a general analytic scheme essentially due to Meinardus (see [25] or [1, Ch. 6]) under which central and local limit theorems will be derived, thus extending Erdős and Lehner's result. The analytic conditions under which we are developing our arguments are weaker than those of Meinardus. Our analytic method can also be applied to the problem left open in [24, p. 311] concerning the common summands in restricted partitions. It turns out that the limiting law is Gaussian for a large class of partitions.

There is another way of counting the number of summands in unrestricted partitions, namely, if the multiplicity of each part is counted only once. Unlike the corresponding counting function (i.e. the $\omega(n)$ function) in the theory of primes (see [38]), this problem is rarely discussed in the theory of partitions. It was first briefly mentioned in [7] in the case $\Lambda = \mathbb{Z}^+$. Wilf [40] introduced the study of distinct components (or sizes of components) in general combinatorial structures. Then Goh and Schmutz [14] derived a central limit theorem for the number of summands for $\Lambda = \mathbb{Z}^+$. The latter result was then extended by Schmutz [34] to multivariate cases under Meinardus's scheme. We further improve and extend their results by establishing the corresponding local limit theorem (in univariate case) under weaker conditions.

A distinctive feature of integer partitions is that the limiting distribution of the number of summands is non-Gaussian in almost all cases if the multiplicity of each summand is taken into account (see [15, 23, 31]), in contrast to the ubiquitous normal law in a large class of combinatorial structures (see [12, 18]). Intuitively, the former phenomenon may be ascribed to the predominance of small summands when the number of summands becomes large, say, larger than the mean value. However, Gaussian limiting distribution appears if the parts are counted without multiplicity, this being intuitively clear since no single part can contribute preponderantly to the corresponding counting function, in accordance with the classical law of errors. Our results show that the same phenomenon still persists if each part is allowed to occur at most once.

For completeness, we add that a formal approach was introduced in Knessel and Keller [21] for characterizing the asymptotic behaviors of many quantities in partition problems satisfying suitable recurrences. Another recent reference on related problems is Fristedt [13], the methods employed there being probabilistic.

We state the main results of this paper in Section 2. The proof of these results is divided into two parts: central (Sect. 3) and local (Sect. 4) limit theorems. In each section, we first derive some necessary estimates and then prove the result in question. Since our assumptions are weaker than those used in [25, 34], some techniques are introduced to justify the regularity conditions (in order to apply the saddle-point method). Unrestricted partitions is treated in Section 5. Finally, we discuss some examples in Section 6 and conclude with some remarks for further extensions.

2. STATEMENT OF RESULTS

Throughout this paper, the symbols c_j always denote absolute positive constants. The symbol ε represents always suitable small quantity whose value may vary from one occurrence to another.

Given a sequence of positive integers $1 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ tending to infinity, let $\Pi(n) = \Pi_\Lambda(n)$ be the set of partitions of the positive integer n into *distinct* parts λ_j (each λ_j occurring at most once), $j = 1, 2, \ldots$ (in the case when there are more than one λ_i with the same value, one can think of properly labeling these λ_i's so that each is "different" from the other). Let $q(n) = |\Pi(n)|$, the cardinality of the set $\Pi(n)$. It is more convenient to work with a_k, denoting the number of λ_j's such that $\lambda_j = k$. The generating function of $q(n)$ satisfies

$$Q(z) = Q_\Lambda(z) = 1 + \sum_{n \geqslant 1} q(n)\, z^n = \prod_{j \geqslant 1} (1 + z^{\lambda_j}) = \prod_{k \geqslant 1} (1 + z^k)^{a_k}, \qquad (1)$$

for $|z| < 1$.

To state our results, we first introduce an analytic scheme essentially due to Meinardus [25] in which the sequence $\{a_k\}$ satisfies the following three conditions.

(M1) The Dirichlet series $D(s) = \sum_{k \geqslant 1} a_k k^{-s}$ converges in the half-plane $\Re e\, s > \alpha > 0$, and can be analytically continued into the half-plane $\Re e\, s \geqslant -\alpha_0$, for some $\alpha_0 > 0$. In $\Re e\, s \geqslant -\alpha_0$, D is analytic except for a simple pole at $s = \alpha$ with residue A[1].

(M2) There exists an absolute constant c_1 such that[2] $D(s) \ll |t|^{c_1}$ uniformly for $\Re e\, s \geqslant -\alpha_0$ as $|t| \to +\infty$.

[1] In Meinardus's original paper, the quantity α_0 is assumed to satisfy $0 < \alpha_0 < 1$.

[2] The Vinogradov symbol $\ll$ is the same as the Landau symbol $O(.)$ and is used interchangeably as is convenient.

(M3) Define $g(\tau)=\sum_{k\geqslant 1} a_k e^{-k\tau}$, where $\tau=r+iy$ with $r>0$ and $-\pi\leqslant y\leqslant\pi$. There exists a positive constant c_2 such that $g(r)-\Re e\, g(\tau)\geqslant c_2(\log(1/r))^{2+4/\alpha^2}$ uniformly for $\pi/2\leqslant |y|\leqslant\pi$ as $r\to 0^+$.

The assumption (M3) here is much weaker than those used in [25] and [34], the essential difference being that we did not impose a similar estimate for $g(r)-\Re e\, g(r+iy)$ in the region $r\leqslant |y|\leqslant \pi/2$, which is established by other assumptions, notably by the growth properties of the sum function $\sum_{k\leqslant X} a_k$ and by (M3).

Introducing a uniform probability measure on the set $\Pi(n)$, we consider the random variable ϖ_n, counting the number of summands in a random partition of n. The bivariate generating function of ϖ_n satisfies

$$Q(u,z)=1+\sum_{n\geqslant 1} q(n)\,\mathrm{E}(u^{\varpi_n})\,z^n=\prod_{k\geqslant 1}(1+uz^k)^{a_k}, \tag{1}$$

for finite u and $|z|<1$, where $\mathrm{E}(u^{\varpi_n})$ represents the probability generating function of ϖ_n.

Set $\kappa=A\Gamma(\alpha)(1-2^{-\alpha})\,\zeta(\alpha+1)$,

$$\mu_n=(\kappa\alpha)^{1/(\alpha+1)}\,\frac{(1-2^{1-\alpha})\,\zeta(\alpha)}{\alpha(1-2^{-\alpha})\,\zeta(\alpha+1)}\,n^{\alpha/(\alpha+1)},$$

$$\sigma_n^2=(\kappa\alpha)^{1/(\alpha+1)}\left(\frac{(1-2^{2-\alpha})\,\zeta(\alpha-1)}{\alpha(1-2^{-\alpha})\,\zeta(\alpha+1)}-\frac{(1-2^{1-\alpha})^2\,\zeta(\alpha)^2}{(\alpha+1)(1-2^{-\alpha})^2\,\zeta(\alpha+1)^2}\right)n^{\alpha/(\alpha+1)}.$$

Here Γ is the Gamma-function, ζ is Riemann's zeta function and the factor $(1-2^{-s})\,\zeta(s+1)$ is defined to be $\log 2$ when $s=0$. Note that $\sigma_n>0$ as can be checked.

THEOREM 1. *Suppose that the sequence* $\{a_k\}$ *satisfies* (M1), (M2), *and* (M3). *Set* $\varpi_n^*=(\varpi_n-\mu_n)/\sigma_n$. *Then the random variable* ϖ_n *is asymptotically normally distributed with mean* $\mathrm{E}(\varpi_n)\sim\mu_n$ *and variance* $\mathrm{Var}(\varpi_n)\sim\sigma_n^2$:

$$\Pr\{\varpi_n^*<x\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t+o(1), \tag{2}$$

uniformly for all x *as* $n\to+\infty$. *Moreover, for sufficiently large* n, *we have the exponential bounds*:

$$\Pr\{\varpi_n^*\geqslant x\}\leqslant\begin{cases} e^{-x^2/2}(1+O((\log n)^{-1})), & \\ \qquad\text{if}\quad 0\leqslant x\leqslant n^{\alpha/(6\alpha+6)}/\log n, & \\ e^{-n^{\alpha/(6\alpha+6)}\,x/(2\log n)}(1+O((\log n)^{-1})), & \\ \qquad\text{if}\quad x\geqslant n^{\alpha/(6\alpha+6)}/(\log n), & \end{cases} \tag{3}$$

and the same inequalities for $\Pr\{\varpi_n^*\leqslant -x\}$.

The method of proof consists of analytic and probabilistic parts: the analytic part is based on Mellin transform and the saddle-point method; and the probabilistic part utilizes Curtiss's theorem [4] for convergence of moment generating functions. It turns out that Lerch's zeta function (see [6, Sect. 1.11]) $\Phi(z, s, v) = \sum_{n \geqslant 0} z^n (n+v)^{-s}$ intervenes in a natural way in our analysis. Our application of the saddle-point method differs from that in [25] and [1, Ch. 6] and yields a better error term. We complete the asymptotic normality of ϖ_n by its strong concentration property (1) using a simple technique (see [27, Ch. III]) amended from the usual Chernoff bound.

We can also derive a local limit theorem in the form of Cramér-type large deviations (see [20] [27]). It suffices to replace condition (M3) by the following stronger one.

(M3') There exists a fixed constant $c_3 > 0$ such that $g(r) - \Re e\, e^{i\theta} g(r+iy) \geqslant c_3 (\log(1/r))^{2+4/\alpha^2}$ uniformly for $\pi/2 \leqslant |y| \leqslant \pi$ and $-\pi \leqslant \theta \leqslant \pi$, as $r \to 0^+$.

Let $Y(u, s)$ be the Mellin transform of the function $\log(1 + ue^{-x})$:

$$Y(u, s) = \int_0^\infty x^{s-1} \log(1 + ue^{-x})\, \mathrm{d}x \qquad \text{for} \quad \Re e\, s > 0. \tag{4}$$

As we will see, Y is essentially Lerch's zeta function.

Theorem 2. *Assume that the sequence* $\{a_k\}$ *satisfies* (M1), (M2) *and* (M3'). *If* $m = \mu_n + x\sigma_n \in \mathbb{Z}^+$, *where* $x = o(n^{\alpha/(2\alpha+2)})$, *then*

$$\Pr\{\varpi_n = m\} = \frac{e^{-x^2/2 + \xi(x/\sigma_n)\, n^{\alpha/(\alpha+1)}}}{\sqrt{2\pi}\, \sigma_n} \left(1 + O\left(\frac{|x|}{n^{\alpha/(2\alpha+2)}} + n^{-\alpha_2/(\alpha+1)}\right)\right), \tag{5}$$

uniformly in x, *where* $\alpha_2 = \min\{1, \alpha_0, \alpha\}$ *and* $\xi(w) = \sum_{j \geqslant 3} \xi_j w^j$ *is analytic in a neighborhood of the origin whose Taylor coefficients satisfy*

$$\xi_k = \frac{-1}{k} [w^{k-2}]\, U''(w) \left(\frac{U'(w) - U'(0)}{U''(0)\, w}\right)^{-k} \qquad \textit{for} \quad k = 3, 4, 5, \ldots \tag{6}$$

with

$$U(w) = (\alpha+1)\, \alpha^{-\alpha/(\alpha+1)} A^{1/(\alpha+1)} \left(Y(e^w, \alpha)^{1/(\alpha+1)} - Y(1, \alpha)^{1/(\alpha+1)} \right).$$

Here the symbol $[z^n]\, f(z)$ *denotes the coefficient of* z^n *in the Taylor expansion of* $f(z)$.

Note that U is convex due to the same property of Y and that $\mu_n = U'(0)\, n^{\alpha/(\alpha+1)}$ and $\sigma_n^2 = U''(0)\, n^{\alpha/(\alpha+1)}$. The first two terms of ξ_k are given by (see [20])

$$\xi_3 = \tfrac{1}{6}\,'''(0) \qquad \text{and} \qquad \xi_4 = \tfrac{1}{24}\left(U^{(4)}(0) - \frac{U'''(0)^2}{U''(0)}\right).$$

As an interesting consequence, we state the following

Corollary 1. *If* $m = \mu_n + x\sigma_n \in \mathbb{Z}^+$, *where* $x = o(n^{\alpha/(6\alpha+6)})$, *then*

$$\Pr\{\varpi_n = m\} = \frac{e^{-x^2/2}}{\sqrt{2\pi}\,\sigma_n}\left(1 + O\left(\frac{|x| + |x|^3}{n^{\alpha/(2\alpha+2)}} + n^{-\alpha_2/(\alpha+1)}\right)\right), \tag{7}$$

uniformly in x.

The proof of this theorem utilizes essentially the two-dimensional saddle-point method and is technically more involved. As is usual in the application of the saddle-point method, it is the verification of the regularity conditions to which much of our analysis is devoted. Actually, we prove more (see Proposition 2 below) but content ourselves with the statement of the theorem.

Our methods can also be applied to the number of distinct parts in unrestricted partitions (repetition allowed) under the same assumptions (M1)–(M3) as in Theorem 1.

Let $\tilde{\Pi}(n)$ represent the set of unrestricted partitions of n and let $p(n)$ be its cardinality. Let ω_n be the number of distinct parts (i.e., counted *without* multiplicities) in a random partition of n, where all $p(n)$ partitions of n are equally likely. The bivariate generating function of ω_n satisfies

$$P(u, z) = 1 + \sum_{n \geqslant 1} p(n)\, \mathrm{E}(u^{\omega_n})\, z^n = \prod_{k \geqslant 1} \left(1 + \frac{uz^k}{1 - z^k}\right)^{a_k}, \tag{8}$$

for $|z| < 1$.

Set $\kappa_1 = A\Gamma(\alpha)\,\zeta(\alpha+1)$,

$$\tilde{\mu}_n = A\Gamma(\alpha)(\kappa_1\alpha)^{-\alpha/(\alpha+1)}\, n^{\alpha/(\alpha+1)} = \frac{(\kappa_1\alpha)^{1/(\alpha+1)}}{\alpha\zeta(\alpha+1)}\, n^{\alpha/(\alpha+1)},$$

$$\tilde{\sigma}_n^2 = \frac{(\kappa_1\alpha)^{1/(\alpha+1)}}{\alpha\zeta(\alpha+1)}\left(1 - 2^{-\alpha} - \frac{\alpha}{(\alpha+1)\,\zeta(\alpha+1)}\right) n^{\alpha/(\alpha+1)}.$$

Define

$$Z(u, s) = \int_0^\infty \log\left(1 + \frac{u}{e^x - 1}\right) x^{s-1}\, dx \qquad \text{for} \quad \Re e\, s > 0 \text{ and } |\arg u| < \pi.$$

It is obvious, by (1), that $Z(u, s) = \Gamma(s)\, \zeta(s+1) + Y(u-1, s)$.

THEOREM 3. *Under the assumptions* (M1), (M2), *and* (M3), *the random variable* ω_n *satisfies asymptotically* $\mathrm{E}(\omega_n) \sim \tilde{\mu}_n$, $\mathrm{Var}(\omega_n) \sim \tilde{\sigma}_n^2$, *and*

$$\Pr\{\omega_n = \tilde{\mu}_n + x\tilde{\sigma}_n\} = \frac{e^{-x^2/2 + \eta(x/\tilde{\sigma}_n)\, n^{\alpha/(\alpha+1)}}}{\sqrt{2\pi}\, \tilde{\sigma}_n} \times \left(1 + O\left(\frac{|x|}{n^{\alpha/(2\alpha+2)}} + n^{-\alpha_2/(\alpha+1)}\right)\right),$$

uniformly for all $x = o(n^{\alpha/(2\alpha+2)})$ *such that* $\tilde{\mu}_n + x\tilde{\sigma}_n \in \mathbb{Z}^+$. *Here* α_2 *is as in Theorem* 2 *and* $\eta(w) = \sum_{j \geqslant 3} \eta_j w^j$ *is analytic at the origin with coefficients given by*

$$\eta_k = \frac{-1}{k}\, [w^{k-2}]\, V''(w) \left(\frac{V'(w) - V'(0)}{V''(0)\, w}\right)^{-k} \qquad \textit{for} \quad k = 3, 4, 5, ...,$$

where

$$V(w) = (\alpha+1)\, \alpha^{-\alpha/(\alpha+1)} A^{1/(\alpha+1)} (Z(e^w, \alpha)^{1/(\alpha+1)} - Z(1, \alpha)^{1/(\alpha+1)}).$$

As the proof of this theorem parallels that of Theorems 1 and 2, only the necessary regularity conditions is worked out in Section 5.

That the assumptions needed for the local limit theorem of ω_n are weaker than those for ϖ_n is seen by the following example. Take $\lambda_j = 2j - 1$. Then it is obvious that the span of the random variable ϖ_n is 2 whereas that of ω_n is 1. More precisely, $\mathrm{E}(u^{\varpi_n})$ contains only odd (respectively even) powers of u for odd (respectively even) n. In this case, local limit theorem of ϖ_n depends on the parity of n.

3. CENTRAL LIMIT THEOREM

3.1. *Lemmas*

In this section, we establish some estimates for the function $Q(u, e^{-\tau})$ (defined in (1)) as $\tau \to 0$. We write consistently the complex variable τ in the form $\tau = r + iy$ with $-\pi \leqslant y \leqslant \pi$ and $r > 0$. These estimates are slightly more general than our need for the proof of Theorem 1 since some of them will be required when establishing the corresponding local limit theorem.

Let $f(u, \tau) = \log Q(u, e^{-\tau})$:

$$f(u, \tau) = \sum_{k \geqslant 1} a_k \log(1 + ue^{-k\tau}).$$

The sum on the right-hand side being a harmonic sum (see [10]), we have available the Mellin inversion formula

$$f(u, \tau) = \frac{1}{2\pi i} \int_{\alpha+1-i\infty}^{\alpha+1+i\infty} D(s)\, Y(u, s)\, \tau^{-s}\, ds, \tag{9}$$

for $\Re e\, \tau > 0$, where $Y(u, s)$ is the Mellin transform of the function $\log(1 + ue^{-x})$; see (1). Note that, for $|u| \leqslant 1$ and $\Re e\, s > 0$, $Y(u, s)$ satisfies

$$Y(u, s) = \Gamma(s) \sum_{j \geqslant 1} \frac{(-1)^{j-1}}{j^{s+1}} u^j, \tag{10}$$

a representation no longer useful when $|u| > 1$. In particular,

$$Y(1, s) = (1 - 2^{-s})\, \zeta(s+1)\, \Gamma(s),$$

so that $\kappa = A\, Y(1, \alpha)$. Now by integration by parts, we see that $Y(u, s)$ is related to the Lerch zeta function $\Phi(z, s, v)$ by

$$Y(u, s) = u\Gamma(s)\, \Phi(-u, s+1, 1),$$

with Φ defined by (see [6])

$$\Phi(z, s, v) = \sum_{k \geqslant 0} \frac{z^k}{(k+v)^s} \qquad \text{for} \quad |z| < 1,\ s \in \mathbb{C}, \text{ and } v \neq 0, -1, -2, \ldots.$$

Analytic properties of $Y(u, s)$ are summarized in the following lemma.

LEMMA 1. *For each fixed u lying in the cut-plane $\mathbb{C} \setminus (-\infty, -1]$, the Mellin transform $Y(u, s)$ can be meromorphically continued into the whole s-plane with simple poles at $s = 0, -1, -2, \ldots$ Moreover, $Y(u, s)$ satisfies the estimate*

$$|Y(u, \sigma + it)| \ll e^{-(\pi/2 - \varepsilon)|t|} \qquad \textit{for any} \quad \varepsilon > 0 \textit{ as } |t| \to +\infty, \tag{11}$$

uniformly for finite σ and u in the cut-plane.

Proof. For completeness, we sketch here a self-contained proof. For further properties, see Erdélyi [6]. First, by integration by parts, we have

$$Y(u, s) = \frac{u}{s} \int_0^\infty \frac{x^s}{e^x + u}\, dx,$$

the right-hand side providing a meromorphic continuation of Y to the half-plane $\Re e\, s > -1$. The first assertion of the lemma follows from repeating the same process. As to (11), since $\log(1 + ue^{-x})$ is an analytic function of x in the half-plane $\Re e\, x > 0$, we have by Cauchy's theorem

$$Y(u, s) = \int_0^{e^{i\varphi}\infty} \log(1 + ue^{-x})\, x^{s-1}\, dx \qquad \text{for any} \quad |\varphi| \leqslant \pi/2 - \varepsilon.$$

Thus a change of variable yields

$$Y(u, s) = e^{i\varphi s} \int_0^\infty \log(1 + ue^{-e^{i\varphi}t})\, t^{s-1}\, dt,$$

from which (11) follows. ∎

Remark 1. Since for $x \sim 0$

$$\log(1 + ue^{-x}) = \log(1 + u) + \sum_{h \geqslant 1} \frac{ux^h}{h!(1+u)^h} \sum_{0 \leqslant j < h} A(h-1, j)(-1)^{h-1-j} u^j,$$

the $A(h, j)$ being Eulerian numbers (see [3, Sect. 6.5]), the residue of $Y(u, s)$ at the simple pole $s = -h$ is equal to $\log(1 + u)$ if $h = 0$; and to

$$\frac{u}{h!\,(1+u)^h} \sum_{0 \leqslant j < h} A(h-1, j)(-1)^{h-1-j} u^j$$

if $h = 1, 2, 3, \ldots$

Lemma 2. *If u lies in the sector $|\arg(u+1)| \leqslant \pi - \varepsilon$, then the function $Q(u, e^{-\tau})$ satisfies*

$$Q(u, e^{-\tau}) = (1+u)^{D(0)}\, e^{AY(u, \alpha)\,\tau^{-\alpha}} (1 + O(|\tau|^{\alpha_1})), \tag{12}$$

uniformly as $\tau \to 0$ in the Stolz angle $|\arg \tau| \leqslant \pi/4$, where $\alpha_1 = \min\{\alpha_0, 1\}$.

Proof. Starting with the integral representation (9), we proceed by shifting the path of integration to the vertical line $\Re e\, s = -\alpha_0$ (using the estimate of D and of Y) and by collecting the residues of the poles encountered,

$$f(u, \tau) = A\, Y(u, \alpha)\, \tau^{-\alpha} + uD(0) \int_0^{\infty} \frac{\mathrm{d}t}{e^t + u} + O(|\tau|^{\alpha_1}),$$

as $\tau \to 0$ in $|\arg \tau| \leqslant \pi/4$. Note that if $\alpha_0 > 1$ then Y has a simple pole at $s = -1$ (provided that $D(-1) \neq 0$), and that if $\alpha_0 = 1$ then the integration path need to be suitably deformed. Using the remark before the lemma, we obtain formula (12). ▮

As is typical in the use of the saddle-point method, we need a uniform estimate for the ratio $|Q(u, e^{-\tau})/Q(u, e^{-r})|$. For that purpose, we first state a result on the growth order of the sum function $\sum_{1 \leqslant k \leqslant X} k^{\ell} a_k$ as X gets large. Besides later applications, the result and the methods are of some independent interest *per se*.

LEMMA 3. *Under* (M1) *the sequence* a_k *satisfies*

$$\sum_{1 \leqslant k \leqslant X} k^{\ell} a_k \sim \frac{A}{\alpha + \ell}\, X^{\alpha + \ell} \qquad (\ell = 0, 1, 2, \ldots), \tag{13}$$

as $X \to +\infty$. *If we further assume* (M2), *then*

$$\sum_{1 \leqslant k \leqslant X} k^{\ell} a_k = \frac{A}{\alpha + \ell}\, X^{\alpha + \ell} (1 + O(X^{-\ell - \alpha/2^L} (\log X)^{L/2^L} + X^{-(\alpha + \alpha_0)/2^L})), \tag{14}$$

where $L = [c_1] + 1$.

Proof. The first formula (7) is a consequence of our assumption on the Dirichlet series $D(s)$ and the Tauberian theorem of Ikehara (see [38, p. 265]). Note that for the validity of (7), the analytic continuation of D to the left of the line of convergence is not required when applying Ikehara's Tauberian theorem. To prove (7), set $F_0(X) = \sum_{1 \leqslant k \leqslant X} a_k k^{\ell}$ and

$$\begin{aligned} F_h(X) &= \sum_{1 \leqslant k \leqslant X} a_k k^{\ell} \left(\log \frac{X}{k} \right)^h \\ &= h \int_1^X \frac{F_{h-1}(t)}{t}\, \mathrm{d}t \qquad (h = 1, 2, 3, \ldots, X \geqslant 1). \end{aligned}$$

Then we have the integral representation (see [38, Ch. II.2])

$$F_L(X) = \frac{1}{2\pi i} \int_{\alpha + \ell + 1 - i\infty}^{\alpha + \ell + 1 + i\infty} \alpha + \ell + 1 \,\frac{X^s}{s^{L+1}}\, D(s - \ell)\, \mathrm{d}s.$$

By (M2), $D(\sigma-\ell+it) \ll |t|^{c_1}$ uniformly in the half-plane $\Re e\, s \geqslant \ell-\alpha_0$. Thus the integral on the right member is absolutely convergent. By Cauchy's theorem, we obtain

$$F_L(X)=\frac{AX^{\alpha+\ell}}{(\alpha+\ell)^{L+1}}+O(R_L(X)), \tag{15}$$

where

$$R_L(X)=(\log X)^L+X^{\ell-\alpha_0}.$$

To describe the asymptotic behavior of F_0, we employ the following differencing argument. Consider first F_{L-1}, which can be written as

$$F_{L-1}(X)=\frac{F_L(X+\delta X)-F_L(X)}{L\log(1+\delta)}-\frac{1}{\log(1+\delta)}\int_X^{X+\delta X}\frac{F_{L-1}(t)-F_{L-1}(X)}{t}\,\mathrm{d}t,$$

for any $\delta>0$. Thus

$$\left|F_{L-1}(X)-\frac{AX^{\alpha+\ell}}{(\alpha+\ell)^{L-1}}\right|\leqslant E_1+E_2\,,$$

where

$$E_1=\left|\frac{F_L(X+\delta X)-F_L(X)}{L\,\log(1+\delta)}-\frac{AX^{\alpha+\ell}}{(\alpha+\ell)^{L-1}}\right|,$$

$$E_2=\frac{1}{\log(1+\delta)}\int_X^{X+\delta X}\frac{F_{L-1}(t)-F_{L-1}(X)}{t}\,\mathrm{d}t.$$

Since $F_{L-1}(t)$ is non-decreasing (the a_k being $\geqslant 0$), we have

$$\begin{aligned}E_2&\leqslant\frac{1}{\log(1+\delta)}\int_X^{X+\delta X}\frac{F_{L-1}(t)}{t}\,\mathrm{d}t-\left(\log\frac{1}{1-\delta}\right)^{-1}\int_{X-\delta X}^{X}\frac{F_{L-1}(t)}{t}\,\mathrm{d}t\\&=\frac{1}{\delta}\,(F_L(X+\delta X)-2F_L(X)+F_L(X-\delta X))\\&\quad+\ O(\max\{F_L(X+\delta X)-F_L(X),\,F_L(X)-F_L(X-\delta X)\}\,),\end{aligned}$$

as $\delta\sim 0$. From (7) and the estimates $(1+\delta)^{\alpha+\ell}-1=(\alpha+\ell)\,\delta+O(\delta^2)$, as $\delta\sim 0$, it follows that

$$E_1+E_2\ll\delta X^{\alpha+\ell}+\frac{R_L(X)}{\delta}\,.$$

Taking $\delta = X^{-(\alpha+\ell)/2} R_L(X)^{1/2}$ $(\to 0^+)$ so as to balance the two error terms on the right-hand side, we obtain

$$F_{L-1}(X) = \frac{AX^{\alpha+\ell}}{(\alpha+\ell)^{L-1}} + O(R_{L-1}(X)),$$

where

$$R_{L-1}(X) = X^{\alpha/2} (\log X)^{L/2} + X^{\ell + (\alpha - \alpha_0)/2}.$$

Repeating the same process, we see that for $j = 1, 2, 3, ..., L$

$$F_{L-j}(X) = \frac{AX^{\alpha+\ell}}{(\alpha+\ell)^{L-j}} + O(R_{L-j}(X)),$$

where

$$R_{L-j}(X) = X^{(1-2^{-j})\alpha} (\log X)^{L/2^j} + X^{\ell + (1-2^{-j})\alpha - \alpha_0/2^j}.$$

This completes the proof. ∎

According to (M2), the number c_1 (and thus L) depends on the value of α_0. One may choose a suitable α_0 (if possible) so that the error terms in (7) are minimized.

An interesting consequence of this lemma is the following

Corollary 8. *As $n \to +\infty$,*

$$\max \varpi_n \sim \alpha^{-1} (\alpha+1)^{\alpha/(\alpha+1)} A^{1/(\alpha+1)} n^{\alpha/(\alpha+1)}, \tag{16}$$

where the max is taken over all distinct partitions of n.

Proof. For, by Lemma 3, if

$$n = \sum_{1 \leqslant k \leqslant X} k a_k \sim \frac{A}{\alpha+1} X^{\alpha+1},$$

then $X \sim ((\alpha+1)\, n/A)^{1/(\alpha+1)}$; consequently,

$$\max \varpi_n = \sum_{1 \leqslant k \leqslant X} a_k \sim \frac{A}{\alpha} X^{\alpha},$$

from which we obtain (16). ∎

LEMMA 4. *Let u be a positive real number. There exists a constant $c_5>0$ such that the inequality*

$$\frac{|Q(u, e^{-r-iy})|}{Q(u, e^{-r})} \leqslant \exp\left(-\frac{c_5 u}{(1+u)^2}(\log(1/r))^2\right) \tag{17}$$

holds uniformly for $r^{1+3\alpha/7} \leqslant |y| \leqslant \pi$ as $r \to 0^+$.

Proof. To start with, we observe that

$$\begin{aligned}\frac{|Q(u, e^{-r-iy})|}{Q(u, e^{-r})} &= \prod_{k\geqslant 1}\left(1-\frac{2ue^{-kr}}{(1+ue^{-kr})^2}(1-\cos ky)\right)^{a_k/2}\\ &\leqslant \exp\left(-\frac{u}{(1+u)^2}\sum_{k\geqslant 1} a_k e^{-kr}(1-\cos ky)\right).\end{aligned} \tag{18}$$

In view of the assumption (M3), it suffices to show that

$$G(r) := g(r) - \Re e\ g(r+iy) \geqslant c_6(\log(1/r))^2, \tag{19}$$

for $r^{1+3\alpha/7} \leqslant |y| \leqslant \pi/2$, as $r \to 0^+$. Consider first the case $r \leqslant |y| \leqslant (\log(1/r))^{-2/\alpha}$. Using the elementary inequality

$$1-\cos t \geqslant \frac{2}{\pi^2}t^2 \qquad \text{for} \quad |t| \leqslant \pi, \tag{20}$$

we have

$$\begin{aligned}G(r) &> \sum_{1\leqslant k\leqslant 1/|y|} a_k e^{-kr}(1-\cos ky) \geqslant \frac{2}{\pi^2} y^2 e^{-r/|y|} \sum_{1\leqslant k\leqslant 1/|y|} k^2 a_k\\ &\geqslant \left(\frac{2Ae^{-1}}{\pi^2(\alpha+2)}-\varepsilon\right)|y|^{-\alpha} \geqslant c_6(\log(1/r))^2 \qquad \text{as} \quad r\to 0^+,\end{aligned}$$

by (13). Next, if $r^{1+3\alpha/7} \leqslant |y| \leqslant r$, then

$$\begin{aligned}G(r) &> \sum_{1\leqslant k\leqslant 1/r} a_k e^{-kr}(1-\cos ky) \geqslant \frac{2}{\pi^2} y^2 e^{-1} \sum_{1\leqslant k\leqslant 1/r} k^2 a_k\\ &\geqslant \frac{Ae^{-1}\pi^{\alpha+2}}{\pi^2(\alpha+2)} y^2 r^{-\alpha-2} \geqslant c_7 r^{-\alpha/7} \geqslant c_6(\log(1/r))^2 \qquad \text{as} \quad r\to 0^+.\end{aligned}$$

Finally, if y lies in the range $0 < |y| \leqslant \pi/2$, then there exists an integer ℓ such that

$$\frac{\pi}{2} \leqslant 2^{\ell} \, |y| \leqslant \pi. \tag{21}$$

From the elementary inequality $1 - \cos\theta \geqslant \frac{1}{4}(1 - \cos 2\theta)$, we obtain by induction $1 - \cos\theta \geqslant 4^{-\ell}(1 - \cos 2^{\ell}\theta)$, and consequently

$$\begin{aligned} G(r) &\geqslant 4^{-\ell} \sum_{k \geqslant 1} a_k e^{-kr}(1 - \cos 2^{\ell} k y) \\ &\geqslant c_2 4^{-\ell} \left(\log \frac{1}{r}\right)^{3 + 4/\alpha^2} \geqslant \frac{c_2}{\pi^2} \, y^2 \left(\log \frac{1}{r}\right)^{3 + 4/\alpha^2}, \end{aligned}$$

in virtue of (M3) and (21). Thus

$$G(r) \geqslant \frac{c_2}{\pi^2} \left(\log \frac{1}{r}\right)^2,$$

for y satisfying $(\log(1/r))^{-2/\alpha} \leqslant |y| \leqslant \pi/2$. Taking $c_5 = \min\{c_2/\pi^2, c_6\}$, (17) follows. ∎

3.2. *Proof of Theorem* 1

Throughout this section, u is a positive real number which eventually will be taken to be near 1.

PROPOSITION 1. *Let $\delta > 0$ be any fixed number in the unit interval. Then we have, uniformly for $\delta \leqslant u \leqslant \delta^{-1}$,*

$$Q_n(u) = \beta(u) \, n^{-(1 + \alpha/2)/(\alpha + 1)} e^{(1 + 1/\alpha) \, K(u) \, n^{\alpha/(\alpha + 1)}} (1 + O_\delta(n^{-\alpha_2/(\alpha + 1)})), \tag{22}$$

the O-term holding uniformly in u, where $\alpha_2 = \min\{\alpha, \alpha_0, 1\}$, $K(u) = (\alpha A \, Y(u, \alpha))^{1/(\alpha + 1)}$ and

$$\beta(u) = (1 + u)^{D(0)} \frac{(\alpha A \, Y(u, \alpha))^{1/(2\alpha + 2)}}{\sqrt{2\pi(\alpha + 1)}} = (1 + u)^{D(0)} \sqrt{\frac{K(u)}{2\pi(\alpha + 1)}}.$$

Proof. By Cauchy's integral formula,

$$Q_n(u) = \frac{e^{nr}}{2\pi} \int_{-\pi}^{\pi} e^{iny} Q(u, e^{-r - iy}) \, \mathrm{d}y = e^{nr}(I_1 + I_2), \tag{23}$$

where I_1 and I_2 represent the integrals $(2\pi)^{-1}\int_{|y|\leqslant r^{1+3\alpha/7}}$ and $(2\pi)^{-1}\int_{r^{1+3\alpha/7}<|y|\leqslant\pi}$, respectively. Here r is chosen so that $\alpha AY(u,\alpha)\,r^{-\alpha-1}=n$, or, equivalently,

$$r=\left(\frac{\alpha AY(u,\alpha)}{n}\right)^{1/(\alpha+1)}>0. \tag{24}$$

We assume that n is sufficiently large so that $r^{1+3\alpha/7}<\pi$. Consider first I_2, which is bounded above by

$$I_2 \ll Q(u, e^{-r})\, e^{-c_9(\log(1/r))^2} \ll e^{AY(u,\alpha)\,r^{-\alpha}-c_{10}(\log n)^2}, \tag{25}$$

in view of (17) and (24).

For I_1 we have by (12) and a change of variables

$$\begin{aligned} I_1 &= (1+u)^{D(0)}\,\frac{r}{2\pi}\int_{-r^{3\alpha/7}}^{r^{3\alpha/7}} e^{inry+AY(u,\alpha)\,r^{-\alpha}(1+iy)^{-\alpha}}(1+O(r^{\alpha_1}))\,\mathrm{d}y \\ &= I_3+I_4, \end{aligned}$$

where

$$I_3=(1+u)^{D(0)}\,\frac{r}{2\pi}\int_{-r^{3\alpha/7}}^{r^{3\alpha/7}} e^{inry+AY(u,\alpha)\,r^{-\alpha}(1+iy)^{-\alpha}}\,\mathrm{d}y,$$

and

$$I_4 \ll r^{1+\alpha_1}\int_{-r^{3\alpha/7}}^{r^{3\alpha/7}} e^{AY(u,\alpha)\,r^{-\alpha}(1+y^2)^{-\alpha/2}}\,\mathrm{d}y.$$

Using the elementary inequality

$$(1+y^2)^{-\alpha/2}\leqslant 1-(1-2^{-\alpha/2})\,y^2 \qquad \text{for} \quad -1\leqslant y\leqslant 1, \tag{26}$$

we obtain

$$\begin{aligned} I_4 &\ll r^{1+\alpha_1}e^{AY(u,\alpha)\,r^{-\alpha}}\int_{-r^{-\alpha/7}}^{r^{-\alpha/7}} e^{-(1-2^{-\alpha/2})\,AY(u,\alpha)\,r^{-\alpha}y^2}\,\mathrm{d}y \\ &\ll r^{1+\alpha_1+\alpha/2}e^{AY(u,\alpha)\,r^{-\alpha}}. \end{aligned} \tag{27}$$

It remains to evaluate I_3. Setting $B=\sqrt{\alpha(\alpha+1)\,AY(u,\alpha)}$ and making the change of variables $v^2=B^2r^{-\alpha}y^2$, we obtain

$$I_3=(1+u)^{D(0)}\,\frac{r^{1+\alpha/2}e^{AY(u,\alpha)\,r^{-\alpha}}}{2\pi B}\int_{-Br^{-\alpha/7}}^{Br^{-\alpha/7}} e^{-v^2/2}T_r(v)\,\mathrm{d}v,$$

where

$$T_r(v) = \exp\left(-\frac{v^2}{\alpha(\alpha+1)} \sum_{j \geqslant 1} \binom{\alpha+j+1}{j+2} \left(\frac{-iv}{B} \right)^j r^{\alpha j/2} \right)$$
$$= 1 + \frac{(\alpha+2)\, iv^3}{6\sqrt{\alpha(\alpha+1)\, AY(u,\alpha)}}\, r^{\alpha/2} + O(v^4 r^\alpha),$$

from which we deduce that

$$I_3 = (1+u)^{D(0)} \frac{r^{1+\alpha/2} e^{AY(u,\alpha)\, r^{-\alpha}}}{\sqrt{2\pi\alpha(\alpha+1)\, AY(u,\alpha)}} (1 + O(r^\alpha)). \tag{28}$$

Collecting our results (23)–(28) yields

$$Q_n(u) = \frac{(1+u)^{D(0)}\, r^{1+\alpha/2} e^{nr + AY(u,\alpha)\, r^{-\alpha}}}{\sqrt{2\pi\alpha(\alpha+1)\, AY(u,\alpha)}} (1 + O(r^{\alpha_2})),$$

uniformly in u. The relation (22) follows from the above formula using (24). ▮

Proof of Theorem 1. (*Asymptotic normality*) Let $M_n(t) = \mathrm{E}(e^{(\varpi_n - \mu_n)\, t/\sigma_n})$, where t is real. Then by (22)

$$M_n(t) = e^{-\mu_n t/\sigma_n} \frac{Q_n(e^{t/\sigma_n})}{Q_n(1)}$$
$$= \left(\frac{e^{t/\sigma_n}+1}{2} \right)^{D(0)} \left(\frac{Y(e^{t/\sigma_n}, \alpha)}{Y(1,\alpha)} \right)^{1/(2\alpha+2)} e^{\phi_n(t)} (1 + O(n^{-\alpha_2/(\alpha+1)})),$$

uniformly in t, where

$$\phi_n(t) = -\frac{\mu_n t}{\sigma_n} + (1+\alpha^{-1})(K(e^{t/\sigma_n}) - K(1))\, n^{\alpha/(\alpha+1)}$$
$$= -\frac{\mu_n t}{\sigma_n} + U(t/\sigma_n)\, n^{\alpha/(\alpha+1)}.$$

Observe that for $\Re e\, s > 0$

$$Y(e^{t/\sigma_n}, s) = Y(1, s) + \int_0^\infty \log\left(1 + \frac{e^{t/\sigma_n} - 1}{e^x + 1} \right) x^{s-1}\, \mathrm{d}x,$$

and that as $n \to +\infty$

$$\int_0^\infty \log\left(1+\frac{e^{t/\sigma_n}-1}{e^x+1}\right) x^{s-1}\,\mathrm{d}x = h_1\,\frac{t}{\sigma_n}+h_2\,\frac{t^2}{\sigma_n^2}+O\left(\frac{|t|^3}{\sigma_n^3}\right),$$

where, in general,

$$h_k = \frac{(-1)^{k-1}}{k}\int_0^\infty \frac{x^{s-1}}{(e^x+1)^k}\,\mathrm{d}x.$$

Note that

$$h_k = \frac{\Gamma(s)}{k!}\sum_{1\leqslant j\leqslant k}\mathbf{s}(k,j)(1-2^{j-s})\,\zeta(s+1-j) \qquad \text{for}\quad k=1,2,3,..., \tag{29}$$

where the $\mathbf{s}(k, j)$ represent the (signed) Stirling numbers of the first kind (see [3, Chap. 5]). Thus, we have

$$M_n(t) = e^{\phi_n(t)}(1+O(n^{-\min\{\alpha/2,\,\alpha_0,\,1\}/(\alpha+1)}+|t|\,n^{-\alpha/(2\alpha+2)})),$$

and

$$K(e^{t/\sigma_n})-K(1) = \frac{K(1)}{(\alpha+1)\,Y(1,\alpha)} \times\left(\frac{h_1 t}{\sigma_n}+\left(2h_2-\frac{\alpha h_1^2}{(\alpha+1)\,Y(1,\alpha)}\right)\frac{t^2}{2\sigma_n^2}+O(|t|^3\,\sigma_n^{-3})\right).$$

Equivalently, this last relation can be written as

$$U\left(\frac{t}{\sigma_n}\right) = U'(0)\,\frac{t}{\sigma_n}+\frac{U''(0)\,t^2}{2\sigma_n^2}+O\left(\frac{|t|^3}{\sigma^3}\right).$$

From these formulae and the relations (by (29))

$$\mu_n n^{-\alpha/(\alpha+1)} = \frac{K(1)\,h_1}{\alpha Y(1,\alpha)}$$

and

$$\sigma_n^2\,n^{-\alpha/(\alpha+1)} = K(1)\left(\frac{2h_2}{\alpha Y(1,\alpha)}-\frac{h_1^2}{(\alpha+1)\,Y(1,\alpha)^2}\right),$$

it follows that

$$M_n(t) = e^{t^2/2}(1 + O(n^{-\alpha_2/(\alpha+1)} + (|t| + |t|^3)\, n^{-\alpha/(2\alpha+2)}))$$
$$= e^{t^2/2}(1 + O(n^{-\min\{\alpha/2,\, \alpha_0,\, 1\}/(\alpha+1)})), \tag{30}$$

uniformly in t. By the theorem of Curtiss in [4], we conclude that the distribution of the random variable ϖ_n is asymptotically Gaussian.

(*Exponential tails*) As to the exponential bounds (3), we observe from the above derivations that (30) remains valid if $|t|$ tends to infinity slowly enough: $t = o(n^{\alpha/(6\alpha+6)})$. We consider only the case when $\varpi_n^* \geqslant x$ in the following, the other case $-\varpi_n^* \geqslant x$ being similar. From (10), we have for $x \geqslant 0$

$$\Pr\{\varpi_n^* \geqslant x\} = \Pr\{e^{\varpi_n^* t} \geqslant e^{tx}\} \leqslant e^{-tx} M_n(t)$$
$$= e^{-tx + t^2/2}(1 + O(n^{-\alpha_2/(\alpha+1)} + (|t| + |t|^3)\, n^{-\alpha/(2\alpha+2)})). \tag{31}$$

Let T be any positive quantity tending to infinity with n and satisfying $T = o(n^{\alpha/(6\alpha+6)})$. If $0 \leqslant x \leqslant T$ then we take (see [27, Chap. III]) $t = x$ in (31) (so as to minimize $-tx + t^2/2$) and we obtain

$$\Pr\{\varpi_n^* \geqslant x\} \leqslant e^{-x^2/2}(1 + O(n^{-\alpha_2/(\alpha+1)} + |T|^3\, n^{-\alpha/(2\alpha+2)}));$$

and if $x \geqslant T$ we have by taking $t = T$:

$$\Pr\{\varpi_n^* \geqslant x\} \leqslant e^{-Tx/2}(1 + O(n^{-\alpha_2/(\alpha+1)} + |T|^3\, n^{-\alpha/(2\alpha+2)})).$$

Now the estimates (1) follow from choosing $T = n^{\alpha/(6\alpha+6)}/\log n$.

(*Mean and variance*) We still have to prove that the mean and the variance of ϖ_n are asymptotic to μ_n and σ_n^2, respectively, a result that is not guaranteed by convergence in distribution. Although we may directly evaluate $Q_u(1, z)$ and $Q''_{uu}(1, z)$ as in the proof of Proposition 1, the asymptotic form of the variance depends on the values of α and α_0. The following arguments are computationally simpler and are not subject to the values of α.

Note that $F_n(x)$, the distribution function of ϖ_n^*, converges pointwise to the standard normal distribution whose mean and variance are 0 and 1, respectively. It suffices to show that $\mathbf{E}\varpi_n^* = o(1)$ and $\mathrm{Var}(\varpi_n^*) = 1 + o(1)$. For the former we use the representation

$$\mathrm{E}(\varpi_n^*) = \int_0^\infty (1 - F_n(x) - F_n(-x))\, \mathrm{d}x,$$

the uniform bounds (3) and Lebesgue's dominated convergence theorem. The latter follows from

$$\mathrm{E}(\varpi_n^{2*}) = \int_0^\infty 2x(1 - F_n(x) - F_n(-x))\,\mathrm{d}x,$$

and similar considerations. ▌

Remark 2. A further refinement of the above arguments leads to the better estimates:

$$\mathrm{E}(\varpi_n) \sim \mu_n + \frac{D(0)}{2} + \frac{(1-2^{1-\alpha})\,\zeta(\alpha)}{2(\alpha+1)(1-2^{-\alpha})\,\zeta(\alpha+1)}, \tag{32}$$

$$\begin{aligned}\mathrm{Var}(\varpi_n) \sim \sigma_n^2 &+ \frac{D(0)}{4} + \frac{\zeta(\alpha-1)(2^\alpha-4)}{(\alpha+1)\,\zeta(\alpha+1)(2^\alpha-1)}\\ &- \frac{\zeta(\alpha)(2^\alpha-2)}{(\alpha+1)\,\zeta(\alpha+1)(2^\alpha-1)}\\ &- \frac{\zeta(\alpha)^2\,(2^\alpha-2)^2}{(\alpha+1)\,\zeta(\alpha+1)^2\,(2^\alpha-1)^2},\end{aligned} \tag{33}$$

where the convention that $(1-2^{-s})\,\zeta(s+1) = \log 2$ when $s=0$ is assumed.

4. LOCAL LIMIT THEOREM

4.1. *Lemmas*

The local behavior of $Q(u, e^{-\tau})$ as $\tau \to 0$ and $u \to 1$ having been made explicit in Lemma 2, we need only consider other ranges of u and τ.

As in the last section, we write consistently

$$u = \rho e^{i\theta} = e^{\varrho + i\theta} \qquad \text{and} \qquad z = e^{-\tau} = e^{-r-iy},$$

where $\rho, r > 0$, $\varrho \in \mathbb{R}$, and $-\pi \leqslant \theta, y \leqslant \pi$. Define

$$G_\theta(r) = g(r) - \Re e\; e^{i\theta} g(r+iy) = \sum_{k \geqslant 1} a_k e^{-kr}(1 - \cos(\theta - ky)),$$

so that $G_0(r) = G(r)$. Then we have, by the derivations for (18),

$$\frac{|Q(\rho e^{i\theta}, e^{-r-iy})|}{Q(\rho, e^{-r})} \leqslant \exp\left(-\frac{\rho G_\theta(r)}{(1+\rho)^2}\right).$$

LEMMA 5. *There exists a positive constant c_{13} such that the inequality*

$$\frac{|Q(\rho e^{i\theta}, e^{-r-iy})|}{Q(\rho, e^{-r})} < \exp\left(-\frac{c_{13}\rho}{(1+\rho)^2}\left(\log\frac{1}{r}\right)^2\right)$$

holds uniformly for $r^{1+3\alpha/7} \leqslant |y| \leqslant \pi$ and $-\pi \leqslant \theta \leqslant \pi$, as $r \to 0^+$.

Proof. Consider first the case when $r \leqslant |y| \leqslant (\log(1/r))^{-2/\alpha}$. If $|\theta| \leqslant \frac{1}{2}$ then

$$\begin{aligned} G_\theta(r) &> \sum_{1/|y| \leqslant k \leqslant 2/|y|} a_k e^{-kr}(1-\cos(\theta - ky)) \\ &> \left(1-\cos\left(\frac{1}{2}\right)\right) e^{-2} \sum_{1/|y| \leqslant k \leqslant 2/|y|} a_k \\ &> c_{14}\,|y|^{-\alpha} \geqslant c_{13}\left(\log\frac{1}{r}\right)^2, \end{aligned}$$

by (13) with $\ell = 0$.

Next, if $\frac{1}{2} \leqslant |\theta| \leqslant \pi$ then

$$\begin{aligned} G_\theta(r) &> \sum_{1/(8|y|) \leqslant k \leqslant 1/(4|y|)} a_k e^{-kr}(1-\cos(\theta - ky)) \\ &> \left(1-\cos\left(\frac{1}{4}\right)\right) e^{-1/4} \sum_{1/(8|y|) \leqslant k \leqslant 1/(4|y|)} a_k \\ &> c_{15}\,|y|^{-\alpha} \geqslant c_{13}\left(\log\frac{1}{r}\right)^2, \end{aligned}$$

again by Lemma 7.

Now consider the case $r^{1+3\alpha/7} \leqslant |y| \leqslant r$ and $-\pi \leqslant \theta \leqslant \pi$. We have

$$\begin{aligned} G_\theta(r) &> \sum_{1/r \leqslant k \leqslant 2/r} a_k e^{-kr}(1-\cos(\theta - ky)) \\ &> (1-\cos(r^{3\alpha/7}))\, e^{-2} \sum_{1/r \leqslant k \leqslant 2/r} a_k \\ &> c_{16} r^{-\alpha/7} > c_{13}\left(\log\frac{1}{r}\right)^2, \end{aligned}$$

by the inequality (20) and Lemma 3.

For the remaining ranges $(\log(1/r))^{-2/\alpha} \leqslant |y| \leqslant \pi$, it suffices, by (M3'), to consider the case $(\log(1/r))^{-2/\alpha} \leqslant |y| < \pi/2$ for which we use the same argument as in the proof of Lemma 9. For $0 < |y| < \pi/2$ there exists an integer

ℓ such that $\pi/2 \leqslant 2^{\ell}|y| \leqslant \pi$. Using the inequality $1-\cos t \geqslant 4^{-\ell}(1-\cos t)$, we obtain

$$G_\theta(r) > 4^{-\ell} \sum_{k \geqslant 1} a_k e^{-kr}(1-\cos(2^{\ell}\theta - 2^{\ell}ky)).$$

Choose integer k such that $2^{\ell}\theta = 2k\pi + \theta'$ where $-\pi \leqslant \theta' \leqslant \pi$. Thus by (M3')

$$G_\theta(r) \geqslant 4^{-\ell} c_3 (\log(1/r))^{2+4/\alpha^2} \geqslant \frac{c_3}{\pi^2}\, y^2 (\log(1/r))^{2+4/\alpha^2}$$

$$\geqslant \frac{c_3}{\pi^2}\,(\log(1/r))^2,$$

for $(\log(1/r))^{-2/\alpha} \leqslant |y| < \pi/2$. This completes the proof. ∎

LEMMA 6. *For $r^{3\alpha/7} \leqslant |\theta| \leqslant \pi$ and $|y| \leqslant r^{1+3\alpha/7}$, the estimate*

$$\frac{|Q(\rho e^{i\theta}, e^{-r-iy})|}{Q(\rho, e^{-r})} < \exp\left(-\frac{c_{17}\rho}{(1+\rho)^2}\, r^{-\alpha/7}\right)$$

holds uniformly in θ and y as $r \to 0^+$.

Proof. We have

$$G_\theta(r) > \sum_{1/(3r) \leqslant k \leqslant 1/(2r)} a_k e^{-kr}(1-\cos(\theta - ky))$$

$$> (1-\cos(\tfrac{1}{2} r^{3\alpha/7}))\, e^{-1/2} \sum_{1/(3r) \leqslant k \leqslant 1/(2r)} a_k$$

$$> c_{18} r^{-\alpha/7},$$

by the inequality (20) and Lemma 3. ∎

We also need the asymptotic behaviors of $Y(u, \alpha)$ as $u \to \infty$ and $u \to 0$, which are described by the following lemma.

LEMMA 7. *The function $Y(u, \alpha)$ satisfies*

$$Y(u, \alpha) = \frac{(\log u)^{\alpha+1}}{\alpha(\alpha+1)}(1 + O((\log u)^{-2})), \tag{34}$$

$$Y'_u(u, \alpha) = \frac{(\log u)^{\alpha}}{\alpha u}(1 + O((\log u)^{-2})), \tag{35}$$

as $|u| \to +\infty$ in the sector $|\arg u| \leqslant \pi - \varepsilon$.

Proof. By the integral representation

$$Y(u, \alpha) = \frac{u}{\alpha} \int_0^\infty \frac{x^\alpha}{e^x + u} \, dx,$$

and the Mellin inversion formula

$$\frac{1}{1+x} = \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\pi}{\sin \pi w} x^{-w} \, dw \qquad (|\arg x| \leqslant \pi - \varepsilon),$$

we obtain

$$Y(u, \alpha) = u\Gamma(\alpha) \frac{1}{2\pi i} \int_{1/2 - i\infty}^{1/2 + i\infty} \frac{\pi u^{-w}}{(1-w)^{\alpha+1} \sin \pi w} \, dw.$$

Deforming the path of integration into a suitable Hankel-type contour in the style of [11], we deduce the result (34). The formula (35) is derived either in a completely analogous manner or by (34) using Ritt's theorem (see [26, pp. 9–11]). ∎

Corollary 3. *The function $U(w)$ satisfies*

$$U'(w) = \begin{cases} \alpha^{-1}(\alpha+1)^{\alpha/(\alpha+1)} A^{1/(\alpha+1)}(1 + (w^{-2})), & \text{as} \quad w \to +\infty; \\ \alpha^{-\alpha/(\alpha+1)}(A\Gamma(\alpha))^{1/(\alpha+1)} e^{w/(\alpha+1)}(1 + O(e^w)), & \text{as} \quad w \to -\infty. \end{cases} \tag{36}$$

Proof. These formulae follow from (35), (10) and the definition of U. ∎

The limiting value of $U'(w)$ as $w \to +\infty$ is a natural one in view of Corollary 2 and the next lemma.

We next consider the solution to the system

$$\begin{cases} n = \alpha A\, Y(e^\varrho, \alpha)\, r^{-\alpha-1} \\ m = A e^\varrho Y'_u(e^\varrho, \alpha)\, r^{-\alpha}, \end{cases} \tag{37}$$

which will be needed when applying the two-dimensional saddle-point method.

For convenience, set

$$M_0 = \alpha^{-1}(\alpha+1)^{\alpha/(\alpha+1)} A^{1/(\alpha+1)} \qquad \text{and} \qquad M_1 = \alpha^{-\alpha/(\alpha+1)}(A\Gamma(\alpha))^{1/(\alpha+1)}.$$

LEMMA 8. *For m lying in the range*

$$1 \leqslant m \leqslant (M_0 - \varepsilon)\, n^{\alpha/(\alpha+1)}, \tag{38}$$

there exists a unique solution (ϱ, r) *to the system* (37) *such that* $r > 0$ *and* $\varrho \in \mathbb{R}$.

Proof. The solution to the first equation of (37) exists for all finite (and real) ϱ and satisfies

$$r = \left(\frac{\alpha A\, Y(e^{\varrho}, \alpha)}{n}\right)^{1/(\alpha+1)} > 0.$$

Substituting this expression into the second equation of (37) yields

$$m = A e^{\varrho} Y'_u(e^{\varrho}, \alpha)(\alpha A\, Y(e^{\varrho}, \alpha))^{-\alpha/(\alpha+1)}\, n^{\alpha/(\alpha+1)} = U'(\varrho)\, n^{\alpha/(\alpha+1)}. \tag{39}$$

Thus there exists a unique real solution to (37) whenever m lies in the range (38). Moreover, if $\varepsilon \to 0^+$ then by (36)

$$\varrho \asymp \varepsilon^{-1/2} \to +\infty \qquad \text{and} \qquad r = \varrho \left(\frac{A}{(\alpha+1)\, n}\right)^{1/(\alpha+1)} (1 + O(\varrho^{-2})).$$

On the other hand, if $m = o(n^{\alpha/(\alpha+1)})$ then by (36)

$$\varrho = (\alpha+1) \log \frac{m}{M_1 n^{\alpha/(\alpha+1)}} \left(1 + O\left(\frac{m}{n^{\alpha/(\alpha+1)}}\right)\right),$$

$$r = \frac{\alpha m}{n}\left(1 + O\left(\frac{m}{n^{\alpha/(\alpha+1)}}\right)\right).$$

In this case, we have $\varrho \to -\infty$ and $r \to 0$. ▮

COROLLARY 4. *If $m = \mu_n + x\sigma_n$, where $x = o(\sigma_n)$, then the solution* (ϱ, r) *satisfies*

$$\begin{aligned} \varrho &= \sum_{j \geqslant 1} \varrho_j \left(\frac{x}{\sigma_n}\right)^j \qquad \textit{and} \\ r &= (\alpha A)^{1/(\alpha+1)}\, n^{-1/(\alpha+1)} \left(Y(1, \alpha)^{1/(\alpha+1)} + \sum_{j \geqslant 1} r_j \left(\frac{x}{\sigma_n}\right)^j \right), \end{aligned} \tag{40}$$

with

$$\varrho_m = \frac{1}{m}\,[w^{m-1}] \left(\frac{U'(w) - U'(0)}{U''(0)\,w}\right)^{-m},$$

$$r_m = \frac{1}{m(\alpha+1)}\,[w^{m-1}]\, e^w Y'_u(e^w, \alpha)\; Y(e^w, \alpha)^{-\alpha/(\alpha+1)} \left(\frac{U'(w) - U'(0)}{U''(0)\,w}\right)^{-m},$$

for $m = 1, 2, 3, \ldots$ *The series are convergent.*

Proof. The relation

$$m = \mu_n + x\sigma_n = U'(0)\, n^{\alpha/(\alpha+1)} + x\sqrt{U''(0)\, n^{\alpha/(\alpha+1)}} \qquad (x = o(\sigma_n))$$

can be written into the more convenient form

$$\frac{U'(\varrho) - U'(0)}{U''(0)} = \frac{x}{\sigma_n},$$

in view of (39). Thus the solution (ϱ, r) satisfies (40) by the Lagrange inversion formula. ∎

4.2. *The Proof of Theorem* 2

Let $q(n, m)$ denote the number of restricted partitions of n having exactly m parts: $q(n, m) = [u^m z^n]\, Q(u, z)$.

PROPOSITION 2. *If* m *lies in the range*

$$m \asymp n^{\alpha/(\alpha+1)} \qquad \text{and} \qquad m \leqslant (M_0 - \varepsilon)\, n^{\alpha/(\alpha+1)},$$

then $q(n, m)$ *satisfies*

$$q(n, m) = \frac{(1 + e^{\varrho})^{D(0)}}{2\pi Bb}\, r^{1+\alpha} e^{-m\varrho + nr + AY(e^{\varrho}, \alpha)\, r^{-\alpha}} (1 + O(r^{\alpha_2})), \qquad (41)$$

where (ϱ, r) *is the unique real solution to the system* (37), $B = \sqrt{\alpha(\alpha+1)\, AY(e^{\varrho}, \alpha)}$ *and*

$$b = \sqrt{Ae^{\varrho} Y'_u(e^{\varrho}, \alpha) + Ae^{2\varrho} Y''_{uu}(e^{\varrho}, \alpha) - \frac{\alpha A e^{2\varrho}}{(\alpha+1)\, Y(e^{\varrho}, \alpha)}\, Y'_u(e^{\varrho}, \alpha)^2}.$$

Proof. We use Cauchy's integral formula

$$q(n, m) = \frac{1}{(2\pi i)^2\, q(n)} \oint_{|u| = \rho} \oint_{|z| = e^{-r}} u^{-m-1} z^{-n-1} Q(u, z)\, \mathrm{d}z\, \mathrm{d}u$$

$$= \frac{e^{-m\varrho + nr}}{4\pi^2 q(n)} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-im\theta + iny} Q(e^{\varrho + i\theta}, e^{-r - iy})\, \mathrm{d}y\, \mathrm{d}\theta,$$

where (ϱ, r) is chosen to satisfy the system (37). Set $r_0 = r^{3\alpha/7}$. The ranges of integration are split into three parts:

$$\begin{array}{rll} \text{(I)} & |\theta| \leqslant r_0, & |y| \leqslant r r_0 \\ \text{(II)} & r_0 < |\theta| \leqslant \pi, & |y| \leqslant r r_0 \\ \text{(III)} & |\theta| \leqslant \pi, & r r_0 < |y| \leqslant \pi. \end{array}$$

By Lemmas 5 and 6, we have

$$\iint_{\text{(II)}} + \iint_{\text{(III)}} \ll \exp\left(-\frac{c_{19} e^{\varrho}}{(1 + e^{\varrho})^2} \left(\log \frac{1}{r} \right)^2 \right) \qquad (c_{19} = \min\{c_{13}, c_{17}\}). \tag{42}$$

It remains to evaluate the integral $\iint_{\text{(I)}}$. By Lemma 2 and a change of variables ($\alpha_1 = \min\{1, \alpha_0\}$)

$$J := \frac{r e^{-m\varrho + nr}}{4\pi^2} \int_{-r_0}^{r_0} \int_{-r_0}^{r_0} (1 + e^{\varrho + i\theta})^{D(0)}$$

$$\times\, e^{-im\theta + inry + AY(e^{\varrho + i\theta}, \alpha)(1 + iy)^{-\alpha} r^{-\alpha}} (1 + O(r^{\alpha_1}))\, \mathrm{d}y\, \mathrm{d}\theta$$

$$=: J_1 + J_2,$$

say, where J_1 corresponds to the main term in the integrand and

$$J_2 \ll e^{-m\varrho + nr} r^{1 + \alpha_1} \int_{-r_0}^{r_0} \int_{-r_0}^{r_0} e^{A\, |Y(e^{\varrho + i\theta}, \alpha)|\, |1 + iy|^{-\alpha} r^{-\alpha}}\, \mathrm{d}y\, \mathrm{d}\theta$$

$$\leqslant\ e^{-m\varrho + nr} r^{1 + \alpha_1} \int_{-r_0}^{r_0} \int_{-r_0}^{r_0} e^{A\, |Y(e^{\varrho + i\theta}, \alpha)|\, (1 - (1 - 2^{-\alpha/2})\, y^2)\, r^{-\alpha}}\, \mathrm{d}y\, \mathrm{d}\theta,$$

where the inequality (26) was used. By the choice of r_0, we have $r_0^2 r^{-\alpha} \to +\infty$. Thus

$$J_2 \ll e^{-m\varrho + nr} r^{1 + \alpha_1 + \alpha/2} \int_{-r_0}^{r_0} e^{A\, |Y(e^{\varrho + i\theta}, \alpha)|\, r^{-\alpha}}\, \mathrm{d}\theta.$$

From the local expansion

$$Y(e^{\varrho+i\theta},\alpha)=Y(e^{\varrho},\alpha)+i\theta\int_0^\infty \frac{e^{\varrho}x^{\alpha-1}}{e^x+e^{\varrho}}\,\mathrm{d}x-\frac{\theta^2}{2}\int_0^\infty \frac{e^{\varrho+x}x^{\alpha-1}}{(e^x+e^{\varrho})^2}\,\mathrm{d}x+O(|\theta|^3),$$

as $\theta\sim 0$, we deduce that $|Y(e^{\varrho+i\theta},\alpha)|\leqslant Y(e^{\varrho},\alpha)-c_{20}\theta^2$ as $r\to 0^+$. It follows that

$$J_2\ll e^{-m\varrho+nr}r^{1+\alpha_1+\alpha}e^{AY(e^{\varrho},\alpha)\,r^{-\alpha}}. \tag{43}$$

We now concentrate on the principal part J_1 for which we introduce the following abbreviations:

$$Y_\theta=Y(e^{\varrho+i\theta},\alpha),\qquad Y_0=Y(e^{\varrho},\alpha),\qquad Y_0'=Y_u'(e^{\varrho},\alpha),$$
$$Y_0''=Y_{uu}''(e^{\varrho},\alpha),\qquad Y_0'''=Y_{uuu}'''(e^{\varrho},\alpha).$$

Consider first the inner integral of J_1:

$$J_3=\frac{e^{AY_\theta r^{-\alpha}}}{2\pi}\int_{-r_0}^{r_0} e^{inry+AY_\theta((1+iy)^{-\alpha}-1)\,r^{-\alpha}}\,\mathrm{d}y.$$

Setting $B=\sqrt{\alpha(\alpha+1)\,AY_0}$ and carrying out the change of variables $y=r^{\alpha/2}v/B$, we obtain

$$J_3=\frac{r^{\alpha/2}e^{AY_\theta r^{-\alpha}}}{2\pi B}\int_{-Br^{-\alpha/7}}^{Br^{-\alpha/7}}\exp\left(\Upsilon iv-\frac{Y_\theta v^2}{2Y_0}+\frac{(\alpha+2)\,Y_\theta}{6BY_0}\,iv^3r^{\alpha/2}+O(r^\alpha v^4)\right)\mathrm{d}v,$$

where $\Upsilon=\alpha Ar^{-\alpha/2}(Y_0-Y_\theta)/B$. Since $v^3r^{\alpha/2}$, $v^4r^\alpha\to 0$ in the range of integration, we have

$$\exp\left(\Upsilon iv-\frac{Y_\theta v^2}{2Y_0}+\frac{(\alpha+2)\,Y_\theta}{6BY_0}\,iv^3r^{\alpha/2}+O(r^\alpha v^4)\right)=e^{\Upsilon iv-v^2/2}(1+R_1),$$

where

$$R_1=-\frac{Y_0'}{2Y_0}\,e^{\varrho}i\theta v^2+\frac{(\alpha+2)\,Y_\theta}{6BY_0}\,iv^3r^{\alpha/2}+O(\theta^2v^2+\theta^2v^4).$$

Substituting this estimate into J_3, we obtain

$$\begin{aligned}J_3&=\frac{r^{\alpha/2}e^{AY_\theta r^{-\alpha}}}{2\pi B}\int_{-Br^{-\alpha/7}}^{Br^{-\alpha/7}} e^{\Upsilon iv-v^2/2}(1+R_1)\,\mathrm{d}v\\&=\frac{r^{\alpha/2}e^{AY_\theta r^{-\alpha}}}{2\pi B}\int_{-\infty}^{\infty} e^{\Upsilon iv-v^2/2}(1+R_1)\,\mathrm{d}v+O\big(r^{9\alpha/14}e^{Ar^{-\alpha}\,\Re\mathrm{e}\,Y_\theta-(1/2)\,B^2r^{-2\alpha/7}}\big).\end{aligned}$$

The integral on the right-hand side can be evaluated by Cauchy's theorem:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\Upsilon iv - v^2/2} v^L \, \mathrm{d}v = \frac{e^{-\Upsilon^2/2}}{\sqrt{2\pi}} \sum_{0 \leqslant 2\ell \leqslant L} \binom{L}{2\ell} \frac{(2\ell)!}{2^\ell \ell!} (i\Upsilon)^{L-2\ell},$$

for any $L = 0, 1, 2, ...$ Thus

$$J_3 = \frac{r^{\alpha/2} e^{A Y_\theta r^{-\alpha} - \Upsilon^2/2}}{\sqrt{2\pi}\, B} (1 + R_2) + O(R_3),$$

where

$$R_2 = \frac{Y_0' e^\varrho}{2 Y_0} i\theta(\Upsilon^2 - 1) + \frac{(\alpha+2)\, Y_\theta}{6 B Y_0} (\Upsilon^3 - 3\Upsilon)\, r^{\alpha/2},$$

$$R_3 = \theta^2 r^{\alpha/2} (|\Upsilon|^2 + |\Upsilon|^4)\, e^{A r^{-\alpha} \Re\mathrm{e}\, Y_\theta - \Re\mathrm{e}\, \Upsilon^2/2} + r^{9\alpha/14} e^{A r^{-\alpha} \Re\mathrm{e}\, Y_\theta - B^2 r^{-2\alpha/7}/2},$$

the error term being meaningful as long as

$$\Upsilon = \frac{\alpha A}{B} r^{-\alpha/2} (Y_0 - Y_\theta) \asymp r^{-\alpha/2} |\theta| \ll r^{-\alpha/7},$$

which is obviously satisfied when $|\theta| \leqslant r_0$.

Returning to J_1, we have

$$J_1 = \frac{r^{1+\alpha/2} e^{-m\varrho + nr}}{2\pi \sqrt{2\pi}\, B} \times \int_{-r_0}^{r_0} (1 + e^{\varrho + i\theta})^{D(0)} \left(e^{im\theta + A Y_\theta r^{-\alpha} - \Upsilon^2/2} (1 + R_2) + O(R_3)\right) \mathrm{d}\theta.$$

Using the expansion

$$e^{im\theta + A Y_\theta r^{-\alpha} - \Upsilon^2/2} = e^{A Y_0 r^{-\alpha} - b\theta^2 r^{-\alpha}/2} (1 + R_4),$$

where

$$R_4 = -\frac{iAe^\varrho \theta^3}{6} r^{-\alpha} \left(Y_0' + 3e^\varrho Y_0'' + e^{2\varrho} Y_0''' - \frac{3\alpha^2 A}{B^2} e^\varrho Y_0'^2 - \frac{3\alpha^2 A}{B^2} e^{2\varrho} Y_0' Y_0'' \right) + O(r^{-\alpha}\theta^4),$$

we deduce, as the evaluations of J_2 and J_3, that

$$J_1 = \frac{(1+e^{\varrho})^{D(0)}\, r^{1+\alpha}}{2\pi Bb} e^{-m\varrho + nr + AY_0 r^{-\alpha}}(1+O(r^{\alpha})). \tag{44}$$

The formula (41) now follows from (42)–(44). ∎

Proof of Theorem 2. Suppose that $m = \mu_n + x\sigma_n \in \mathbb{Z}^+$, where $x = o(\sigma_n)$. From Proposition 1, we have

$$q(n) = \frac{2^{D(0)}(\alpha A Y(1,\alpha))^{1/(2\alpha+2)}}{\sqrt{2\pi\alpha(\alpha+1)}} n^{-(1+\alpha/2)/(\alpha+1)} e^{U(0)\, n^{\alpha/(\alpha+1)}}(1+O(n^{-\alpha_2/(\alpha+1)})).$$

It follows, by (41), that

$$\Pr\{\varpi_n = m\} = \frac{q(n,m)}{q(n)} = L_n(\varrho)\, e^{H_n(\varrho)}(1+O(n^{-\alpha_2})),$$

where

$$L_n(\varrho) = \frac{(1+e^{\varrho})^{D(0)}\sqrt{\alpha+1}}{2^{D(0)}\sqrt{2\pi}\, Bb(\alpha A Y(1,\alpha))^{1/(2\alpha+2)}} r^{1+\alpha} n^{(1+\alpha/2)/(\alpha+1)},$$

and

$$\begin{aligned} H_n(\varrho) &= -m\varrho + nr + AY(e^{\varrho},\alpha)\, r^{-\alpha} - \frac{\alpha+1}{\alpha}(\alpha A Y(1,\alpha))^{1/(\alpha+1)}\, n^{\alpha/(\alpha+1)} \\ &= n^{\alpha/(\alpha+1)}(U(\varrho) - U(0) - \varrho U'(\varrho)). \end{aligned}$$

By Lagrange inversion formula, we have

$$U(\varrho) - U(0) - \varrho U'(\varrho) = -\frac{U''(0)}{2\sigma_n^2} x^2 + \xi(x/\sigma_n).$$

The desired result (5) now follows from expanding $L_n(\varrho)$ at $\varrho = 0$,

$$\begin{aligned} L_n(\varrho) &= \frac{(\alpha A Y(1,\alpha))^{-\alpha/(2\alpha+2)}\, n^{-\alpha/(2\alpha+2)}}{\sqrt{2\pi\left(AY'_u(1,\alpha) + AY''_{uu}(1,\alpha) - \dfrac{\alpha A}{(\alpha+1)\, Y(1,\alpha)} Y'_u(1,\alpha)^2\right)}}(1+O(\varrho)) \\ &= \frac{1}{\sqrt{2\pi}\,\sigma_n}\left(1 + O\left(\frac{|x|}{\sigma_n}\right)\right), \end{aligned}$$

since

$$U''(0) = A(\alpha A Y(1, \alpha))^{-\alpha/(\alpha+1)} \times \left(A Y'_u(1, \alpha) + A Y''_{uu}(1, \alpha) - \frac{\alpha A}{(\alpha+1)\, Y(1, \alpha)}\, Y'_u(1, \alpha)^2 \right).$$

This completes the proof of Theorem 2. ∎

5. UNRESTRICTED PARTITIONS

Recall that each of the $p(n)$ unrestricted partitions of n (into parts λ_j) is assumed to be equally likely, and that the random variable ω_n represents the number of distinct parts in a random partition of n. The bivariate generating function of ω_n satisfies (8). By the equation

$$\prod_{j \geqslant 1} \left(1 + \frac{uz^j}{1-z^j}\right)^{a_j} = \prod_{j \geqslant 1} \left(\frac{1+(u-1)\, z^j}{1-z^j}\right)^{a_j},$$

we obtain the relation for the generating polynomials $P_n(u) := p(n)\, \mathrm{E}(u^{\omega_n})$ and $Q_n(u) = q(n)\, \mathrm{E}(u^{\varpi_n})$:

$$P_n(u) = \sum_{0 \leqslant j \leqslant n} p(j)\, Q_{n-j}(u-1) \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

To prove Theorem 3, we proceed along the same line of arguments as in the last section. The analytic properties we need are summarized in Propositions 3 and 4 below. The remaining analysis being parallel to the proof of Theorem 2, we omit the details.

Recall that $Z(u, s) = \Gamma(s)\, \zeta(s+1) + Y(u-1, s)$.

Proposition 3. *Let $u \in \mathbb{C}$, $|\arg u| \leqslant \pi - \varepsilon$, where $\varepsilon > 0$ being arbitrarily small but fixed number. If $\tau \to 0$ in the sector $|\arg \tau| \leqslant \pi/4$, then*

$$P(u, e^{-\tau}) = e^{D'(0)} u^{D(0)} \tau^{-D(0)} e^{AZ(u, \alpha)\, \tau^{-\alpha}} (1 + O(|\tau|^{\alpha_1})), \tag{45}$$

uniformly in τ and u, where $\alpha_1 = \min\{1, \alpha_0\}$.

Proof. (Sketch) We have

$$\log P(u, e^{-\tau}) = \sum_{k \geqslant 1} a_k \log\left(1 + \frac{u}{e^{k\tau}-1}\right) = \frac{1}{2\pi i} \int_{\alpha+1-i\infty}^{\alpha+1+i\infty} D(s)\, Z(u, s)\, \tau^{-s}\, \mathrm{d}s.$$

The expansion (45) is obtained by shifting the path of integration to the line $\Re s = -\alpha_0$ and by computing the residues of the poles encountered (see [1, 25, 34]). ∎

To derive uniform estimates for the ratio $|P(\rho e^{i\theta}, e^{-r-iy})|/P(\rho, e^{-r})$, we use the the following inequalities.

LEMMA 9. *Let $u = \rho e^{i\theta}$, where $\rho > 0$ and $|\theta| \leqslant \pi$. Then the inequalities*

$$\frac{|P(\rho e^{i\theta}, e^{-r-iy})|}{P(\rho, e^{-r})} \leqslant e^{-c_{21} T(r)} \tag{46}$$

and

$$\frac{|P(\rho e^{i\theta}, e^{-r-iy})|}{P(\rho, e^{-r})} < e^{-c_{22} G(3r)} \tag{47}$$

hold for $|y| \leqslant \pi$ and $r > 0$, where $c_{21} = \min\{\rho, \rho^{-1}\}$, $c_{22} = \frac{1-e^{-4}}{4} \min\{1, \rho^2\}$, $G(r) = G_0(r)$ and

$$T(r) = \sum_{k \geqslant 1} a_k e^{-kr} (1 - e^{-kr})^2 \, (1 - \cos(\theta - ky)).$$

Proof. First, we have

$$\begin{aligned}\left|1 + \frac{\rho e^{i\theta}}{e^{kr+iky} - 1}\right|^2 &\leqslant 1 + 2\,\Re \frac{\rho e^{i\theta}}{e^{kr+iky}} + \frac{2\,|\rho e^{i\theta}|}{|e^{kr+iky}(e^{kr+iky} - 1)|} + \left|\frac{\rho e^{i\theta}}{e^{kr+iky} - 1}\right|^2 \\ &\leqslant 1 - \frac{2\rho}{e^{kr}} (1 - \cos(\theta - ky)) + \frac{2\rho}{e^{kr} - 1} + \frac{e^{2\rho}}{(e^{kr} - 1)^2} \\ &= \left(1 + \frac{\rho}{e^{kr} - 1}\right)^2 \left(1 - \frac{2\rho(1 - \cos(\theta - ky))}{e^{kr}(1 + \rho/(e^{kr} - 1))^2}\right).\end{aligned}$$

Thus

$$\begin{aligned}\frac{|P(\rho e^{i\theta}, e^{-r-iy})|}{P(\rho, e^{-r})} &\leqslant \prod_{k \geqslant 1} \left(1 - \frac{2\rho(1 - \cos(\theta - ky))}{e^{kr}(1 + \rho/(e^{kr} - 1))^2}\right)^{a_k/2} \\ &\leqslant \exp\left(-\rho \sum_{k \geqslant 1} a_k e^{-kr} \left(1 + \frac{\rho}{e^{kr} - 1}\right)^{-2} (1 - \cos(\theta - ky))\right).\end{aligned}$$

From the inequalities

$$\left(1+\frac{\rho}{e^{kr}-1}\right)^{-2} \geqslant \begin{cases}(1-e^{-kr})^2, & \text{if } \ 0<\rho\leqslant 1;\\ \rho^{-2}(1-e^{-kr})^2, & \text{if } \ \rho\geqslant 1,\end{cases}$$

the result (46) follows.

Next, we have

$$\begin{aligned}\left|1+\frac{\rho e^{i\theta}}{e^{kr+iky}-1}\right|^2 &\leqslant \left(1+\frac{\rho}{|e^{kr+iky}-1|}\right)^2\\ &< \left(1+\frac{\rho}{e^{kr}-1}\right)^2-\left(\frac{\rho^2}{(e^{kr}-1)^2}-\frac{\rho^2}{|e^{kr+iky}-1|^2}\right)\\ &= \left(1+\frac{\rho}{e^{kr}-1}\right)^2\left(1-\frac{\rho^2}{(e^{kr}+\rho-1)^2}\left(1-\frac{(e^{kr}-1)^2}{|e^{kr+iky}-1|^2}\right)\right).\end{aligned}$$

Using the inequalities (see [17, Eq. (3.14)])

$$\left(1+\frac{4vX}{(X-1)^2}\right)^{-1}\leqslant e^{-4v/X} \qquad (0\leqslant v\leqslant 1<X),$$

$$2e^{-kr}(1-\cos ky)\leqslant 4 \qquad (r>0,\ k=1, 2, 3, ...),$$

$$1-e^{-w}\geqslant \frac{1}{4}(1-e^{-4})\, w \qquad (0\leqslant w\leqslant 4),$$

we obtain

$$\begin{aligned}1-\frac{(e^{kr}-1)^2}{|e^{kr+iky}-1|^2} &= 1-\left(1+\frac{2e^{kr}}{(e^{kr}-1)^2}(1-\cos ky)\right)^{-1}\\ &\geqslant 1-\exp(-2e^{-kr}(1-\cos ky))\\ &\geqslant \frac{1}{2}(1-e^{-4})\, e^{-kr}(1-\cos ky).\end{aligned}$$

Thus

$$\begin{aligned}\left|1+\frac{\rho e^{i\theta}}{e^{kr+iky}-1}\right|^2 &< \left(1+\frac{\rho}{e^{kr}-1}\right)^2\left(1-\frac{(1-e^{-4})\,\rho^2 e^{-kr}}{2(e^{kr}+\rho-1)^2}(1-\cos ky)\right)\\ &\leqslant \left(1+\frac{\rho}{e^{kr}-1}\right)^2\\ &\quad\times\left(1-\frac{(1-e^{-4})\min\{1, \rho^2\}}{2}\, e^{-3kr}(1-\cos ky)\right),\end{aligned}$$

in virtue of the inequalities

$$(e^{kr}+\rho-1)^2 \leqslant \begin{cases} e^{2kr}, & \text{if} \quad 0<\rho\leqslant 1; \\ \rho^2 e^{2kr}, & \text{if} \quad \rho\geqslant 1. \end{cases}$$

It follows that

$$\frac{|P(\rho e^{i\theta}, e^{-r-iy})|}{P(\rho, e^{-r})} < \prod_{k\geqslant 1} (1-2c_{22}e^{-3kr}(1-\cos ky))^{a_k/2},$$

from which we derive (47). ∎

Proposition 4. *As $r\to 0^+$, the inequality*

$$\frac{|P(\rho e^{i\theta}, e^{-r-iy})|}{P(\rho, e^{-r})} < \exp\left(-c_{23}\left(\log\frac{1}{r}\right)^2\right)$$

holds for (i) $r^{1+3\alpha/7}\leqslant |y|\leqslant \pi$ *and* $-\pi\leqslant\theta\leqslant\pi$; *and* (ii) $r^{3\alpha/7}\leqslant|\theta|\leqslant\pi$ *and* $|y|\leqslant r^{1+3\alpha/7}$. *Here* c_{23} *can be taken as*

$$c_{23}=\min\{c_{21}c_{18}(1-e^{-1/3})^2, \tfrac{1}{2}c_{22}\min\{c_2, c_6\}\}.$$

Proof. The result in the range (i) is a direct consequence of (19), (47) and Lemma 4 if we assume (M1) and (M3). For the second range we argue as the proof of Lemma 6. We have for θ and y in range (ii)

$$\begin{aligned} T(r) &> \sum_{1/(3r)\leqslant k\leqslant 1/(2r)} a_k e^{-kr}(1-e^{-kr})^2\,(1-\cos(\theta-ky)) \\ &> \left(1-\cos\left(\frac{1}{2}r^{3\alpha/7}\right)\right)e^{-1/2}(1-e^{-1/3})^2 \sum_{1/(3r)\leqslant k\leqslant 1/(2r)} a_k \\ &> c_{24}r^{-\alpha/7} > c_{24}\left(\log\frac{1}{r}\right)^2, \end{aligned}$$

by the inequality (20) and Lemma 3, where $c_{24}=c_{18}(1-e^{-1/3})^2$. From (46), we obtain the required inequality. ∎

Finally, the asymptotic behaviors of $V'(w)$ depend more on the values of α as described in the following result.

Lemma 10. *If* $w\to+\infty$ *then* $V'(w)$ *satisfies*

$$V'(w)=\alpha^{-1}(\alpha+1)^{\alpha/(\alpha+1)}A^{1/(\alpha+1)}(1+O(w^{-2}+w^{-\alpha-1}));$$

and if $w \to -\infty$ *then*

$$V'(w) = \begin{cases} \left(\dfrac{A\pi}{\sin \pi\alpha}\right)^{1/(\alpha+1)} e^{\alpha w/(\alpha+1)}(1+O(e^{(1-\alpha)w})), & \text{if } 0<\alpha<1; \\ -A^{1/2}w(1-w)^{-1/2}\, e^{w/2}(1+O(e^w)), & \text{if } \alpha=1; \\ \alpha^{-\alpha/(\alpha+1)}(A\Gamma(\alpha)\,\zeta(\alpha))^{1/(\alpha+1)} & \\ \qquad \times e^{w/(\alpha+1)}(1+O(e^w+e^{(\alpha-1)w})), & \text{if } \alpha>1. \end{cases}$$

Proof. These follow from the definition of Z and properties of $\Phi(z, s, v)$ (see [6, Sect. 1.11]). ∎

Note that $U'(w) \sim V'(w)$ as $w \to +\infty$, this being intuitively clear in view of (16) and the relation $\max \omega_n \sim \max \varpi_n$.

6. EXAMPLES

In general, it is the condition (M3) or (M3') that is more difficult to check. A sufficient condition for the validity of these two is the following condition of Haselgrove and Temperley (see [15, 31]): there exists a positive constant $\vartheta<1$ such that

$$|g(r+iy)| < \vartheta g(r) \qquad \text{for} \quad r \leqslant |y| \leqslant \pi, \tag{48}$$

as $r \to 0^+$. This condition is satisfied, for example, when $\lambda_j = j^\ell$ (see [15]), where ℓ is a positive integer. However, as remarked by Richmond [31], (48) is, in general, a difficult condition to work with. Sometimes it is easier to check the following condition:

If the abscissa of convergence α_φ *of the Dirichlet series* $D_\varphi(s) = \sum_{k\geqslant 1} a_k e^{ik\varphi} k^{-s}$ *is* $<\alpha$ *for each* $\pi/2 \leqslant |\varphi| \leqslant \pi$, *then* (M3') (*and* a fortiori (M3)) *is satisfied.*

For, by Mellin inversion formula, we deduce $g(r+iy) \ll r^{-\alpha+\varepsilon}$ for $\pi/2 \leqslant |y| \leqslant \pi$, where $\alpha_y < \alpha - \varepsilon < \alpha$. Thus $g(r) - |g(r+iy)| \gg r^{-\alpha}$.

(a) Let $\lambda_j = j^\ell$ for $j = 1, 2, 3, \ldots$, where ℓ is a fixed positive integer. All our theorems apply. Further computations show that the mean and variance of ϖ_n satisfy

$$\mathrm{E}(\varpi_n) = \mu_n + c_{25} + O(n^{-1/(\ell+1)}) \quad \text{and} \quad \mathrm{Var}(\varpi_n) = \sigma_n^2 + c_{26} + O(n^{-1/(\ell+1)}),$$

where expressions for the two constants c_{25} and c_{26} are given in (32) and (33) (with α there replaced by $1/\ell$). In particular, if $\ell = 1$ we have

$$E(\varpi_n) = \frac{2\sqrt{3}\log 2}{\pi}\sqrt{n} + \frac{3\log 2}{\pi^2} - \frac{1}{4} + O(n^{-1/2}),$$

$$\mathrm{Var}(\varpi_n) = \left(\frac{\sqrt{3}}{\pi} - \frac{12\sqrt{3}}{\pi^3}(\log 2)^2\right)\sqrt{n} + \frac{3}{2\pi^2} - \frac{1}{8} - \frac{36(\log 2)^2}{\pi^4} - \frac{3\log 2}{\pi^2} + O(n^{-1/2}),$$

which improve an old result of Erdős and Lehner in [7].

Likewise, we have

$$E(\omega_n) = \tilde{\mu}_n + c_{27} + O(n^{-1/(\ell+1)}) \quad \text{and} \quad \mathrm{Var}(\omega_n) = \tilde{\sigma}_n^2 + c_{28} + O(n^{-1/(\ell+1)}),$$

where

$$c_{27} = -\frac{1}{2} + \frac{\ell}{(\ell+1)\,\zeta(1+1/\ell)} \qquad \text{and}$$

$$c_{28} = \frac{\ell}{(\ell+1)\,\zeta(1+1/\ell)}\,(1 - 2^{-1/\ell} - \zeta(1+1/\ell)^{-1}).$$

(b) Let $\lambda_j = j^\ell - 1$, ℓ being a fixed integer $\geqslant 2$ and $j = 2, 3, 4, \ldots$ We have (see [10, p. 45])

$$D(s) = \sum_{j\geqslant 2} (j^\ell - 1)^{-s} = \sum_{j\geqslant 0} \binom{s+j-1}{j} (\zeta(\ell s + \ell j) - 1),$$

the last expression providing a meromorphic continuation of D into the whole s-plane with polynomial growth order at $\sigma \pm i\infty$. To check (M3), we argue as in [23, Example (c), pp. 39–40]. For $r > 0$ and $0 < |y| \leqslant 1$

$$g(r) - \Re\, g(r + 2\pi i y) \geqslant \sum_{\substack{j\geqslant 2 \\ \cos 2\pi(j^\ell - 1)\,y \leqslant 0}} e^{-(j^\ell - 1)\,r} \geqslant e^{-1/2} \sum_{\substack{2\leqslant j\leqslant r^{-1/\ell} \\ \cos 2\pi(j^\ell - 1)\,y\leqslant 0}} 1. \tag{49}$$

By the Weyl criterion (see [22, p. 7]) the sequence $((j^\ell - 1)\,y)_{j\geqslant 2}$ is uniformly distributed mod 1 for irrational y. Thus for any $\varepsilon > 0$ the number of summands in the rightmost summation is at least $(\frac{1}{2} - \varepsilon)\,r^{-1/\ell}$ for sufficiently small r. It follows that

$$g(r) - \Re\, g(r + 2\pi i y) \geqslant \tfrac{1}{3}\, e^{-1/2} r^{-1/\ell},$$

for irrational y as $r \to 0^+$. But $g(r + 2\pi i y)$ is a continuous function of y (actually infinitely differentiable). Thus condition (M3) holds for y in the interval $\frac{1}{4} \leqslant y \leqslant \frac{3}{4}$. Theorem 1 applies. Similarly, it can be verified that other results are also applicable.

(c) Let $\lambda_j = [j^\beta]$, where $\beta > 1$ is not an integer. We have

$$D(s) = \sum_{j \geqslant 1} [j^\beta]^{-s} = \zeta(\beta s) + \sum_{m \geqslant 1} \binom{s+m-1}{m} \zeta_m(s),$$

where $\zeta_m(s) = \sum_{j \geqslant 1} \{j^\beta\}^m j^{-\beta(s+m)}$ for $\Re s > \beta^{-1} - m$. Here $\{t\}$ denotes the fractional part of t. Thus D admits meromorphic continuation into the half-plane $\Re s > \beta^{-1} - 1 > -1$ with a simple pole at $s = 1/\beta$. Further analytic properties of D can be derived through those of $\zeta_m(s)$. Conditions (M3) and (M3') can be checked as in the last example, the uniform distribution modulo 1 being, for example, a consequence of Weyl's metric theorem (see [22, p. 32]) and the uniform continuity of g.

(d) Let $\lambda_j = h + jd$ with $(h, d) = 1$, h, d being positive integers, $d \geqslant 2$ and $1 \leqslant h < d$. This is an interesting case where the condition (M3) holds (see [25]) but (M3') is violated. Thus Theorems 1 and 3 apply but Theorem 2 does not. This example describes an interesting "period-hereditary" property of the sequence $\{\lambda_j\}$ on ϖ_n, namely, if the sequence $\{\lambda_j\}$ is periodic[3], then ϖ_n is of maximum span > 1. On the other hand, the random variables ω_n are less sensitive to such a property.

(e) $\lambda_{j + \ell(\ell-1)/2} = \ell$, $j = 1, ..., \ell$, namely, $a_k = k$ and $D(s) = \zeta(s-1)$. All our theorems again apply.

7. EXTENSIONS

The results in this paper are susceptible of many different extensions. We only discuss two typical cases in this section.

First, let $\lambda_j = 2^{j-1}$. It is easy to show the relation

$$\prod_{j \geqslant 0} (1 + uz^{2^j}) = 1 + \sum_{k \geqslant 1} u^{\nu(k)} z^k,$$

where $\nu(k)$ denotes the number of 1's in the binary representation of k. In this case $D(s) = (1 - 2^{-s})^{-1}$ with a simple pole at $s = 0$. Despite this degenerate case, it might be possible to extend our results to the case $\alpha = 0$, as suggested by the example $D(s) = 1 + 2^{-s}/(1 - 2^{-s})^2$. On the other hand, a limiting Gaussian law for ω_n can be established using the methods of this

[3] If there exist two integers c and d such that $\lambda_j = c + dj$ with $d > 1$, then the sequence $\{\lambda_j\}$ is called periodic; otherwise, it is called aperiodic.

paper, the regularity conditions requiring somewhat different arguments (see [5]). For further information on Mahler-type partitions, see [29, 35]. Note that if

$$P_n(u) = [z^n] \prod_{j \geqslant 0} \left(1 + \frac{uz^{2^j}}{1 - z^{2^j}}\right) \qquad (n = 0, 1, 2, ...),$$

then

$$\begin{cases} P_{2m}(u) = P_{2m-1}(u) + P_m(u) \\ P_{2m+1}(u) = P_{2m-1}(u) + uP_m(u) \end{cases} \qquad (m = 1, 2, 3, ...),$$

with $P_0(u) = 1$ and $P_1(u) = u$. These relations are useful from a computational point of view.

A natural question suggested by the above examples is that between the degenerate limiting behavior of ϖ_n and the limiting Gaussian behavior of ω_n, from which point on will the "phase change" (from a discrete limiting law to a continuous one) occur? More precisely, let $q_\ell(n)$ denote the number of partitions into parts 2^{j-1} in which each part is allowed to appear at most ℓ-times. Define $\varpi_n^{[\ell]}$ by

$$1 + \sum_{n \geqslant 1} z^n q_\ell(n) \, (u^{\varpi_n^{[\ell]}}) = \prod_{j \geqslant 0} (1 + u(z^{2^j} + z^{2 \cdot 2^j} + \cdots + z^{\ell \cdot 2^j})).$$

The question is for what values of ℓ will the distribution of $\varpi_n^{[\ell]}$ be asymptotically normal, an intuition being that the asymptotic normality of $\varpi_n^{[\ell]}$ would imply the same property for $\varpi_n^{[\ell+1]}$?

Next, take $D(s) = \sum_{p \text{ prime}} p^{-s}$. Numerical evidence suggests again that the limiting distributions of ϖ_n and of ω_n will still be Gaussian. For results on the total number of partitions and the moments of the summands, see Roth and Szekeres [33] and Richmond [30].

As mentioned in the Introduction, the limiting distributions of the number of summands (counted with multiplicities) in unrestricted partitions are non-Gaussian for almost all partitions. However, it was predicted by Haselgrove and Temperley [15] that Gaussian law would appear if $\alpha \geqslant 2$, although a formal proof is still lacking. A concrete example is the generating function

$$\prod_{j \geqslant 1} (1 - uz^j)^{-j},$$

enumerating the number of plane partitions of n with a given sum of the diagonal parts or with a given trace (see [1, Chap. 11]).

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