AN ASYMPTOTIC FORMULA IN THE THEORY OF PARTITIONS

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[Received 18 January 1950]

1. Introduction

THE purpose of this paper is to investigate the asymptotic behaviour, as $k \to \infty$, $n \to \infty$, of the following functions in number theory:

$p(n,k)$, the number of partitions of n into exactly k positive integral parts;

$P(n,k) = \sum_{r<k} p(n,r)$;

$q(n,k)$, the number of partitions of n into exactly k positive unequal parts.

We do not need to consider each one of these functions separately. It follows from the identities

$$P(n,k) = p(n+k,k) = q\left\{n+\binom{k+1}{2},k\right\} \tag{1.1}$$

that once we have an asymptotic expression for, say, $P(n,k)$, then a trivial change in the variables will immediately furnish an expression for $p(n,k)$ or $q(n,k)$, and conversely. Actually we shall find it more convenient to use a fourth arithmetical function $R(N,k)$ defined by

$$R(N,k) = P\{N-\tfrac{1}{4}k(k+1),k\}$$

whose generating function can be managed more easily. From the asymptotic expression for $R(N,k)$ we obtain the corresponding expressions for $P(n,k)$, $p(n,k)$, $q(n,k)$ by putting

$$N = n+\tfrac{1}{4}k(k+1), \qquad N = n+\tfrac{1}{4}k(k-3), \qquad N = n-\tfrac{1}{4}k(k+1)$$

respectively.

The problem of this paper was considered previously by Erdős and Lehner,* who investigated $P(n,k)$ in the neighbourhood of $k = \frac{1}{2}cn^{\frac{1}{2}}\log n$, $c = 6^{\frac{1}{2}}\pi^{-1}$, which is the 'normal' number of summands in the partitions of n. They found that

$$P(n,k)/P(n) = \{1+o(1)\}\exp(-ce^{-\lambda}) \tag{1.2}$$

* P. Erdős and J. Lehner, *Duke Math. Journ.* 8 (1941), 335–45.

Quart. J. Math. Oxford (2), 2 (1951), 85–108

for $k = \frac{1}{2}cn^{\frac{1}{2}}\log n + c\lambda n^{\frac{1}{2}}$, where $P(n) = P(n,n)$ is the number of unrestricted partitions of n. Using their method, Auluck, Chowla, and Gupta* proved

$$n^{\frac{1}{2}}p(n,k)/P(n) = \{1+o(1)\}\exp(-\lambda-ce^{-\lambda})$$

for $k = \frac{1}{2}cn^{\frac{1}{2}}\log n + c\lambda n^{\frac{1}{2}}$. Erdős and Lehner† also investigated $P(n,k)$ for small values of k and proved that

$$P(n,k),\, p(n,k),\, q(n,k) \cong \frac{1}{k!}\binom{n-1}{k-1} \tag{1.3}$$

uniformly for $k = o(n^{\frac{1}{3}})$. This result can be obtained easily from the elementary inequalities

$$\frac{1}{k!}\binom{n-\binom{k}{2}-1}{k-1} \leqslant q(n,k) \leqslant \frac{1}{k!}\binom{n-1}{k-1} \leqslant p(n,k) \leqslant \frac{1}{k!}\binom{n+\binom{k}{2}-1}{k-1}$$

and
$$\frac{1}{k!}\binom{n+k-1}{k-1} \leqslant P(n,k) \leqslant \frac{1}{k!}\binom{n+\binom{k+1}{2}-1}{k-1},$$

which follow from the recursive formula

$$q(n,k) = \sum_{r=1}^{[n/k]} q(n-rk, k-1). \tag{1.4}$$

To prove, for example,

$$q(n,k) \leqslant \frac{1}{k!}\binom{n-1}{k-1},$$

assume induction on k:

$$q(n,k) \leqslant \frac{1}{(k-1)!}\sum_{r>0}\binom{n-rk-1}{k-2} \leqslant \frac{1}{k!}\sum_{r>0}\binom{n-r-1}{k-2} = \frac{1}{k!}\binom{n-1}{k-1}.$$

The first inequality for $q(n,k)$ can be proved similarly, and the inequalities for $p(n,k)$ and $P(n,k)$ follow at once from (1.1).‡

In the present paper I shall obtain an asymptotic formula for $R(N,k)$ which holds in a much wider range than (1.3), namely for $N > Ck^2$, where the constant C has the value 0·385 approximately.§ More precisely, I shall prove the following theorem:

* F. C. Auluck, S. Chowla, and H. Gupta, *J. Indian Math. Soc.* 6 (1942), 105–12.

† P. Erdős and J. Lehner, *Duke Math. Journ.* 8 (1941), 335–45.

‡ Essentially the same proof was given by H. Gupta for $p(n,k)$, *Proc. Indian Acad. Sci.* (A) 16 (1942), 101–2, and by F. C. Auluck for $P(n,k)$, *J. Indian Math. Soc.* 6 (1942), 113–14.

§ The best possible value for C is 0·25, since $N \geqslant \frac{1}{2}k(k+1)$.

THEOREM 1. *Let*

$$u = u(t) = t+\tfrac{1}{36}t^3+\tfrac{41}{32400}t^5+\dots \quad (|t|<4)$$

be defined as the inverse function of

$$t = u^2\left(\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx\right)^{-1} = \left(\tfrac{1}{4}+u^{-2}\int_0^u \frac{x}{e^x-1}\,dx\right)^{-1}, \tag{1.5}$$

and let $\rho = 2{\cdot}598\dots$ *be the radius of convergence of* $v = v(t)$ *defined by*

$$t = v^2\left(\int_0^v (2-\tfrac{1}{2}x\cot\tfrac{1}{2}x)\,dx\right)^{-1}.$$

Write $\alpha = (k+\frac{1}{2})^2/N$. *Then, for every* $m>0$,

$$\log R(N,k) = k\left\{2\frac{u(\alpha)}{\alpha}-\log\{2\sinh\tfrac{1}{2}u(\alpha)\}\right\}-\log N+ \\ +\sum_{\mu=0}^{m-1}k^{-\mu}\psi_\mu(\alpha)+O(k^{-m}) \tag{1.6}$$

uniformly for $\alpha \leqslant \rho_0 < \rho$, *where*

$$\psi_0(t) = \frac{u(t)}{t}-\tfrac{1}{2}\log\left(\frac{2}{u(t)}\sinh\tfrac{1}{2}u(t)\right)+\tfrac{1}{2}\log u'(t)-\log(2\pi) \tag{1.6$'$}$$

and the functions $\psi_\mu(t)$ *are analytic for* $|t|<\rho$.

Although I can prove (1.6) only for $\alpha \leqslant \rho_0 < 2{\cdot}598\dots$, and hence for $N = (k+\frac{1}{2})^2/\alpha \geqslant 0{\cdot}385k^2$, it seems very likely that it holds uniformly for every $N \geqslant Ck^2$ with $C > \frac{1}{4}$. The following argument suggests that (1.6) is valid even in the critical neighbourhood of $C = \frac{1}{4}$ (or $\alpha = 4$), at least for $m = 1$ if $O(k^{-1})$ is replaced by $o(1)$. For, let us make this assumption and take the function $P(n,k)$ at $k = n$. We have

$$\alpha = (k+\tfrac{1}{2})^2/N = (k+\tfrac{1}{2})^2\{k+\tfrac{1}{4}k(k+1)\}^{-1};$$

hence, from (1.5) at $t = \alpha$,

$$u^{-2}\int_0^u \frac{x}{e^x-1}\,dx = \frac{1}{\alpha}-\frac{1}{4} = (k-\tfrac{1}{16})(k+\tfrac{1}{2})^{-2},$$

$$u = c^{-1}(k+\tfrac{1}{2})(k-\tfrac{1}{16})^{-\frac{1}{2}}\{(1+o(k^{-2})\},$$

and

$$2\frac{u}{\alpha}-\log(2\sinh\tfrac{1}{2}u) = 2u\left(\frac{1}{\alpha}-\frac{1}{4}\right)-\log(1-e^{-u}) \\ = 2c^{-1}(k+\tfrac{1}{2})^{-1}(k-\tfrac{1}{16})^{\frac{1}{2}}\{1+o(k^{-2})\}.$$

Also, from (1.5),

$$\frac{du}{dt} = ut^{-2}\left(\frac{2}{t}-\tfrac{1}{2}\coth\tfrac{1}{2}u\right)^{-1} = ut^{-2}\left\{2\left(\frac{1}{t}-\frac{1}{4}\right)-\frac{1}{e^u-1}\right\}^{-1}, \tag{1.7}$$

and hence $$u'(\alpha) = \tfrac{1}{32}c^{-1}k^{\frac{1}{2}}\{1+o(k^{-1})\}.$$

Putting these in (1.6), (1.6′), we easily obtain

$$P(n,n) = \{1+o(1)\}4^{-1}3^{-\frac{1}{2}}n^{-1}\exp(2c^{-1}n^{\frac{1}{2}}),$$

which is the asymptotic term of the Hardy–Ramanujan formula. By a somewhat lengthier computation we can also derive (1.2) from the formula if we assume its validity in the region $\alpha = 4-\delta/\log^2 k$, $\delta > 0$.

The formula (1.6) remains valid if $\alpha \to 0$ as $k \to \infty$: that is, if $k = o(n^{\frac{1}{2}})$. We then have

$$u(\alpha)/\alpha \to 1, \quad u'(\alpha) \to 1,$$

and

$$\log R(N,k) = k\left\{2u(\alpha)/\alpha - \log\left(\frac{2}{\alpha}\sinh\tfrac{1}{2}u(\alpha)\right)\right\} - \log(2\pi N\alpha^k)+1+o(1)$$

$$= k(2-\tfrac{1}{72}\alpha^2+\tfrac{7}{10800}\alpha^4-\dots)+(k-1)\log N-2k\log k-\log(2\pi)+o(1).$$

Notably, if $k = O(n^{\frac{1}{2}-\delta})$, $\delta > 0$, then, for $q(n,k)$,

$$(k-1)\log N = (k-1)\log n+(k-1)\log\{1-\tfrac{1}{4}k(k+1)/n\}$$

$$= (k-1)\log n-(\tfrac{1}{4}k^3/n+\tfrac{1}{32}k^5/n^2+\tfrac{1}{192}k^7/n^3+\dots)+o(1).$$

Hence

$$q(n,k) \cong n^{k-1}\{k!(k-1)!\}^{-1}\exp\{-(\tfrac{1}{4}k^3/n+\tfrac{13}{288}k^5/n^2+\tfrac{1}{192}k^7/n^3+\dots)\}$$

by Stirling's formula. Similarly

$$p(n,k),\ P(n,k) \cong n^{k-1}\{k!(k-1)!\}^{-1}\exp(\tfrac{1}{4}k^3/n-\tfrac{13}{288}k^5/n^2+\tfrac{1}{192}k^7/n^3-\dots).$$

Finally, if $k = O(n^{\frac{1}{3}})$, then

$$q(n,k) \cong n^{k-1}\{k!(k-1)!\}^{-1}\exp(-\tfrac{1}{4}k^3/n)$$

and $$p(n,k),\ P(n,k) \cong n^{k-1}\{k!(k-1)!\}^{-1}\exp(\tfrac{1}{4}k^3/n).$$

Generally, it follows from (1.6) that $p(n,k)$ and $P(n,k)$ are asymptotically equal if $k = o(n^{\frac{1}{2}})$. This was proved by Erdős and Lehner for $k = o(n^{\frac{1}{3}})$ and conjectured for $k = o(n^{\frac{1}{2}})$.

2. A conjecture of Auluck, Chowla, and Gupta

For a fixed n, $p(n,k)$ has a maximum at a certain $k = k_0(n)$. Erdős proved* that

$$k_0 = \tfrac{1}{2}cn^{\frac{1}{2}}\log n+c\log cn^{\frac{1}{2}}+o(n^{\frac{1}{2}}) \quad (c = 6^{\frac{1}{2}}/\pi). \tag{2.1}$$

He also stated that, if $q(n,k)$ has its maximum (for a fixed n) at k_0, then

$$k_0 = (2^{\frac{1}{2}}\log 2)cn^{\frac{1}{2}}+dn^{\frac{1}{4}}+o(n^{\frac{1}{4}}) \tag{2.2}$$

for a certain constant d which he did not specify. Auluck, Chowla, and

* P. Erdős, *Bull. American Math. Soc.* 52 (1946), 185–8.

Gupta conjectured* that $p(n,k)$ is a *monotonous* function of k: that is, $p(n,k_1) < p(n,k_2)$ if $k_1 < k_2 < k_0$ and $p(n,k_1) > p(n,k_2)$ if $k_0 < k_1 < k_2$. I shall now show that the analogous conjecture for $q(n,k)$ is true if n is sufficiently large and $0{\cdot}635k^2 \leqslant n$.†

For a fixed n,

$$\alpha = \alpha_k = (k+\tfrac{1}{2})^2\{n+\tfrac{1}{16}-\tfrac{1}{4}(k+\tfrac{1}{2})^2\}^{-1}$$

is a function of k. We have

$$\Delta\alpha_k = \alpha_k-\alpha_{k-1} = \tfrac{1}{2}\beta(\beta+4)k^{-1}+O(k^{-3}), \tag{2.3}$$

where

$$\beta = k^2(n+\tfrac{1}{16}-\tfrac{1}{4}k^2)^{-1}.$$

Also, if $\phi(t)$ is analytic at $t=\beta$, then

$$\begin{aligned}\Delta\phi(\alpha_k) = \phi(\alpha_k)-\phi(\alpha_{k-1}) &= \phi'(\beta)\Delta\alpha_k+O\{(\Delta\alpha_k)^3\}\\ &= \beta(\beta+4)\phi'(\beta)/2k+O(k^{-3})\end{aligned}$$

and

$$\phi(\alpha_k)+\phi(\alpha_{k-1}) = 2\phi(\beta)+O(k^{-2}).$$

From (1.6) and (1.6′) we have, if

$$\alpha = N^{-1}(k+\tfrac{1}{2})^2 = \{n-\tfrac{1}{4}k(k+1)\}^{-1}(k+\tfrac{1}{2})^2 \leqslant \rho_0 < 2{\cdot}598..., \; n \geqslant 0{\cdot}635k^2,$$

$$\begin{aligned}\log q(n,k) = (k+\tfrac{1}{2})[2u(\alpha_k)/\alpha_k-\log\{2\sinh\tfrac{1}{2}u(\alpha_k)\}]+&\\ +\tfrac{1}{2}\log\{uu'(\alpha_k)\}+\log\alpha_k-\log(k+\tfrac{1}{2})^2-\log(2\pi)+k^{-1}\psi_1(\alpha_k)+O(k^{-2}),&\end{aligned}$$

$$\begin{aligned}&\log q(n,k)-\log q(n,k-1)\\ &= k\left(2\frac{u'}{\beta}-2\frac{u}{\beta^2}-\frac{u'}{2}\coth\tfrac{1}{2}u\right)\Delta\alpha_k+2\frac{u}{\beta}-\log(2\sinh\tfrac{1}{2}u)+\\ &\qquad +\frac{1}{2}\left(\frac{u'}{u}+\frac{u''}{u'}\right)\Delta\alpha_k+\frac{1}{\beta}\Delta\alpha_k-\frac{2}{k}+\frac{1}{k}\psi_1'(\beta)\Delta\alpha_k+O(k^{-2})\\ &= \beta(\beta+4)\left(\frac{u'}{\beta}-\frac{u}{\beta^2}-\frac{u'}{4}\coth\tfrac{1}{2}u\right)+2\frac{u}{\beta}-\log(2\sinh\tfrac{1}{2}u)+\\ &\qquad +\left(\frac{u'}{u}+\frac{u''}{u'}\right)\frac{\beta(\beta+4)}{4k}+\frac{\beta+4}{2k}-\frac{2}{k}+O(k^{-2}),\end{aligned}$$

where u, u', u'' are to be taken at the place β. If we introduce the expression (1.7) for u', we obtain

$$\Delta\log q(n,k) = -\log(e^u-1)+\frac{\beta(\beta+4)}{4k}\left(\frac{u'}{u}+\frac{u''}{u'}\right)+\frac{\beta}{2k}+O(k^{-2}). \tag{2.4}$$

* F. C. Auluck, S. Chowla, and H. Gupta, *J. Indian Math. Soc.* 6 (1942), 105–12.

† These restrictions on n and k are not really important. The first condition (that n be large) can probably be discarded if we replace the O-notations throughout the proof of Theorem 1 by explicit constants and check the conjecture for smaller values of n. But, if we know that the conjecture is true for every n, k satisfying, say, $0{\cdot}9k^2 \leqslant n$, then its validity for $k^2 > n$ can easily be established by induction on n, using the recursion (1.4).

This cannot be zero unless

$$\log\{e^{u(\beta)}-1\} = O(k^{-1}),$$

i.e.

$$u(\beta) = \log 2+O(k^{-1}),$$

$$\frac{1}{\beta}-\frac{1}{4} = u^{-2}\int_0^u \frac{x}{e^x-1}\,dx = \tfrac{1}{2}\{(c\log 2)^{-2}-1\}+O(k^{-1}),$$

$$\beta = \{(2c^2\log^2 2)^{-1}-\tfrac{1}{4}\}^{-1}+O(k^{-1}),$$

since

$$\int_0^{\log 2} \frac{x}{e^x-1}\,dx = \tfrac{1}{2}(\tfrac{1}{6}\pi^2-\log^2 2) = \tfrac{1}{2}(c^{-2}-\log^2 2).*$$

Note that $\{(2c^2\log^2 2)^{-1}-\frac{1}{4}\}^{-1} < 0{\cdot}7$, and hence β is well within the region of validity of (1.6).

From (1.7) we have

$$\frac{u'}{u}(t)+\frac{u''}{u'}(t) = \left\{2(1-\tfrac{1}{4}t)-\frac{t}{e^u-1}\right\}^{-1}\left\{1+\frac{2}{e^u-1}-\frac{te^u}{(e^u-1)^2}u'\right\}.$$

Hence for $t=\beta$,

$$u' = \log 2\{2\beta(1-\tfrac{3}{4}\beta)\}^{-1}+O(k^{-1}),$$

$$\frac{u'}{u}+\frac{u''}{u'} = (2-\tfrac{3}{2}\beta)^{-1}\{3-(1-\tfrac{3}{4}\beta)^{-1}\log 2\}+O(k^{-1}).$$

Therefore the right-hand side of (2.4) cannot be zero unless

$$u(\beta) = \log\left(2+\frac{b}{k}\right)+O(k^{-2})$$

with

$$b = \tfrac{1}{4}\beta(\beta+4)\left(\frac{u'}{u}+\frac{u''}{u'}\right)+\tfrac{1}{2}\beta+O(k^{-1}) = \frac{2}{a-1}-\frac{a\log 2}{2(a-1)^2},$$

$$a = \tfrac{1}{12}(\pi/\log 2)^2. \tag{2.5}$$

If $u(\beta) < \log(2+b/k)$, the right-hand side of (2.4) is certainly positive unless

$$u(\beta) = \log(2+b/k)+O(k^{-2}),$$

i.e.

$$\beta^{-1} = \tfrac{1}{4}+\left(\log 2+\frac{b}{2k}\right)^{-2}\int_0^{\log 2+b/2k} \frac{x}{e^x-1}\,dx+O(k^{-2})$$

$$= a-\tfrac{1}{4}-b(a-1)(k\log 2)^{-1}+O(k^{-2}). \tag{2.6}$$

Similarly (2.4) is negative for $u(\beta) > \log(2+b/k)$ unless (2.6) holds.

* See D. Bierens de Haan, *Nouvelles Tables d'Intégrales définies* (New York, 1939), 151, formula (104, 5).

Suppose now that β satisfies (2.6) and let k_0 be defined by

$$\beta = (k_0+\tfrac{1}{2})^2\{n+\tfrac{1}{16}-\tfrac{1}{4}(k_0+\tfrac{1}{2})^2\}^{-1}. \tag{2.7}$$

Then, if $k_1 \leqslant k_0$ and $\alpha_1 = k_1^2(n+\frac{1}{16}-\frac{1}{4}k_1^2)^{-1}$, we have

$$(\alpha_1)^{-1} \geqslant k_0^{-2}(n+\tfrac{1}{16}-\tfrac{1}{4}k_0^2) \geqslant (1+1/k_0)/\beta,$$

which shows that $(\alpha_1)^{-1}$ does not satisfy (2.6) if k_0 is large. Hence $\Delta \log q(n, k_1) > 0$. Similarly we can show that, if $k_2 \geqslant k_0+1$, then $\Delta \log q(n, k_2) < 0$. This proves the conjecture for large values of $n \geqslant 0{\cdot}635k^2$.

From (2.6) and (2.7) we obtain for the maximum k_0

$$(n+\tfrac{1}{16})(k_0+\tfrac{1}{2})^{-2}-\tfrac{1}{4} = a-\tfrac{1}{4}-b(a-1)\{(k_0+\tfrac{1}{2})\log 2\}^{-1}+O(k_0^{-2}),$$

$$n = a(k_0+\tfrac{1}{2})^2-b(a-1)(k_0+\tfrac{1}{2})/\log 2+O(1),$$

$$k_0 = a^{-\frac{1}{2}}n^{\frac{1}{2}}+b\left(1-\frac{1}{a}\right)\Big/2\log 2-\tfrac{1}{2}+O(n^{-\frac{1}{2}})$$

$$= 0{\cdot}764304...n^{\frac{1}{2}}-0{\cdot}008428...+O(n^{-\frac{1}{2}}). \tag{2.8}$$

This improves the result of Erdős and shows that the constant d in (2.2) is zero. The error term in (2.8) tends to zero. Hence the position of the maximum is exactly known for large values of n.

It is interesting to compare the value of k_0 calculated from (2.8) with data obtained from a table of Todd's* containing all the values of $p(n, k)$ for $n \leqslant 100$. We obtain for $n = 100$, calculated from the table by means of $q(n, k) = p\left\{n-\binom{k}{2}, k\right\}$,

$$q(100, 6) = 65827, \qquad q(100, 7) = 108869,$$

$$q(100, 8) = 116263, \qquad q(100, 9) = 79403,$$

whereas (2.8) gives $k_0 \sim 7{\cdot}63$. But even for small values of n the formula gives the correct value of k_0. For $n = 16$,

$$q(16, 2) = 7, \qquad q(16, 3) = 14, \qquad q(16, 4) = 9,$$

whereas (2.8) gives $k_0 \sim 3{\cdot}05$.

Another result which follows easily from my formula is an asymptotic expression for $Q(n)$, the total number of partitions of n into unequal parts. The function

$$Q(n, k) = \sum_{r<k} q(n, r)$$

is not connected in a simple manner with $R(N, k)$; we therefore have to use the formula

$$Q(n) = \sum_{k} q(n, k). \tag{2.9}$$

* J. A. Todd, *Proc. London Math. Soc.* (2) 48 (1945), 229–42.

Let us first consider those terms which belong to k's in the neighbourhood of k_0.

Write $u(\alpha_k) = \{1+\lambda(k)\}\log 2$, $\lambda = \lambda(k) = O(n^{-\frac{1}{5}})$, and put

$$a = \tfrac{1}{12}(\pi/\log 2)^2 = 1{\cdot}71\ldots.$$

By an easy calculation

$$\frac{1}{\alpha} = \frac{1}{\alpha_k} = \frac{1}{4}+u^{-2}\int\limits_0^u \frac{x}{e^x-1}\,dx$$

$$= a-\tfrac{1}{2}-2(a-1)\lambda+(3a-3-\log 2)\lambda^2+O(\lambda^3),$$

$$k+\tfrac{1}{2} = (n+\tfrac{1}{16})^{\frac{1}{2}}\left(\frac{1}{\alpha}+\frac{1}{4}\right)^{-\frac{1}{2}}$$

$$= (n+\tfrac{1}{16})^{\frac{1}{2}}a^{-\frac{1}{2}}\left\{1+\left(1-\frac{1}{a}\right)\lambda+\frac{1}{2a}\left(\frac{3}{a}-3+\log 2\right)\lambda^2+O(\lambda^3)\right\},$$

$$2u(\alpha)/\alpha-\log\{2\sinh\tfrac{1}{2}u(\alpha)\}$$

$$= 2\log 2\{a-(a-1)\lambda+(a-1-\tfrac{1}{2}\log 2)\lambda^2+O(\lambda^3)\},$$

$$(k+\tfrac{1}{2})\left(\frac{2u}{\alpha}-\log(2\sinh\tfrac{1}{2}u)\right) = 3^{-\frac{1}{2}}\pi(n+\tfrac{1}{16})^{\frac{1}{2}}\left\{1-\frac{1}{2a}\left(1-\frac{1}{a}\right)\lambda^2+O(\lambda^3)\right\},$$

$$\{uu'(\alpha)\}^{\frac{1}{2}} = (a-\tfrac{1}{4})\{2(a-1)\}^{-\frac{1}{2}}\log 2+O(\lambda),$$

$$N = (n+\tfrac{1}{16})\left(1-\frac{1}{4a}\right)\{1+O(\lambda)\}. \tag{2.10}$$

Hence

$$q(n,k) = \frac{1}{2\pi N}(uu')^{-\frac{1}{2}}\exp\left\{(k+\tfrac{1}{2})\left(2\frac{u}{\alpha}-\log(2\sinh\tfrac{1}{2}u)\right)\right\}\{1+O(k^{-1})\}$$

$$= \frac{a\log 2}{2\pi n}\{2(a-1)\}^{-\frac{1}{2}}\exp\left\{3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}\left(1-\frac{1}{2a}\left(1-\frac{1}{a}\right)\lambda^2\right)\right\}\{1+o(1)\}.$$

The condition $\lambda = O(n^{-\frac{1}{5}})$ is certainly satisfied if $k_1 < k < k_2$, where $k_1 = a^{-\frac{1}{2}}n^{\frac{1}{2}}-n^{\frac{3}{10}}$, $k_2 = a^{-\frac{1}{2}}n^{\frac{1}{2}}+n^{\frac{3}{10}}$. Therefore

$$\sum_{k_1<k<k_2} q(n,k) = \{1+o(1)\}\frac{a\log 2}{2\pi n}\{2(a-1)\}^{-\frac{1}{2}}\exp(3^{-\frac{1}{2}}\pi n^{\frac{1}{2}})\times$$

$$\times\sum_{k_1<k<k_2}\exp\left[-3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}\frac{1}{2a}\left(1-\frac{1}{a}\right)\{\lambda(k)\}^2\right]$$

$$= \{1+o(1)\}\frac{a\log 2}{2\pi n}\{2(a-1)\}^{-\frac{1}{2}}a^{-\frac{1}{2}}\left(1-\frac{1}{a}\right)n^{\frac{1}{2}}\times$$

$$\times\int\limits_{\lambda(k_1)}^{\lambda(k_2)}\exp\left\{-3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}\frac{1}{2a}\left(1-\frac{1}{a}\right)\lambda^2\right\}d\lambda,$$

since $\qquad \Delta k = (k+\frac{1}{2})-(k-\frac{1}{2}) = \{1+o(1)\}a^{-\frac{1}{2}}\left(1-\frac{1}{a}\right)n^{\frac{1}{2}}\,d\lambda,$

by (2.10). Writing

$$y = \left\{3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}\frac{1}{2a}\left(1-\frac{1}{a}\right)\right\}^{\frac{1}{2}}\lambda,$$

and, noting that, for $k = k_1$ and $k = k_2$, $|\lambda(k)| > C_1 n^{-\frac{1}{5}}$ for a certain positive constant C_1, and hence that $n^{\frac{1}{2}}\{\lambda(k_i)\}^2 \to \infty$ ($i = 1, 2$), we obtain

$$\sum_{k_1<k<k_2} q(n,k) = \{1+o(1)\}\tfrac{1}{2}a^{\frac{1}{2}}\log 2\pi^{-\frac{1}{2}}n^{-\frac{3}{4}}3^{\frac{1}{4}}\int_{-\infty}^{\infty} e^{-y^2}\,dy\,\exp(3^{-\frac{1}{2}}\pi n^{\frac{1}{2}})$$
$$= \{1+o(1)\}4^{-1}3^{-\frac{1}{4}}n^{-\frac{3}{4}}\exp(3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}).$$

Now it is easily seen that the terms with $k \leqslant k_1$ and $k \geqslant k_2$ in (2.9) do not contribute essentially to the value of $Q(n)$, for $n^{\frac{1}{2}}\lambda^2(k_i) > C_1^2 n^{1/10}$ where $i = 1, 2$. Hence

$$\exp\left[-3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}\frac{1}{2a}\left(1-\frac{1}{a}\right)\{\lambda(k_i)\}^2\right] < \exp(-C_2 n^{1/10})$$

for a positive constant C_2, and

$$q(n,k_i) < q(n,k_0)\exp(-C_2 n^{1/10}),$$
$$\sum_{k<k_1} q(n,k)+\sum_{k>k_2} q(n,k) < n^{\frac{1}{2}}\{q(n,k_1)+q(n,k_2)\} = o\{q(n,k_0)\},$$

which implies $\qquad Q(n) \cong 4^{-1}3^{-\frac{1}{4}}n^{-\frac{3}{4}}\exp(3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}).$

This is the asymptotic term of the Hardy–Ramanujan formula for $Q(n)$. The above proof is entirely independent of the theorem of residues and the transformation properties of elliptic modular functions, which seem to be indispensable for the proof of the exact formula of Hardy–Ramanujan–Hua.*

For $q(n,k_0) = \max q(n,k)$ we obtain

$$q(n,k_0) \cong \tfrac{1}{4}6^{-\frac{1}{2}}\left(1-\frac{1}{a}\right)^{-\frac{1}{2}}n^{-1}\exp(3^{-\frac{1}{2}}\pi n^{\frac{1}{2}}).$$

Hence $\qquad Q(n)/n^{\frac{1}{4}}q(n,k_0) \to 3^{\frac{1}{4}}2^{\frac{1}{2}}(1-1/a)^{\frac{1}{2}} = 1{\cdot}200\ldots.$

Except for the explicit value of the constant on the right, this result was also found by Erdős.†

If we assume the validity of (1.6), (1.6′) at $\alpha = 4-\delta/\log^2 k$ ($\delta > 0$)

* L. K. Hua, *Trans. American Math. Soc.* 51 (1942), 194–201. There is a misprint in the main formula on p. 195 of the paper. $\frac{2}{3}(n+\frac{1}{24})$ should read $\frac{1}{3}(n+\frac{1}{24})$.

† P. Erdős, *Bull. American Math. Soc.* 52 (1946), 185–8.

and do not care about the error terms, then also (2.1) can be obtained from the formula. It seems likely that

$$k_0 = \tfrac{1}{2}cn^{\frac{1}{2}}\log n + c\log cn^{\frac{1}{2}} + d + o(1) \sim 0{\cdot}38985\ldots n^{\frac{1}{2}}\log n - 0{\cdot}19403n^{\frac{1}{2}} + d$$

for a certain constant d. Data taken from Todd's table suggests that the value of d must be near to 1·7.

3. Proof of the main theorem

The proof of Theorem 1 is based on two results. One is the classical theory of 'waves' of Sylvester.* Let ζ denote a primitive qth root of unity ($q \leqslant k$) and write $P_\zeta(n,k)$ for the coefficient of t^{-1} in

$$f(\zeta,t) = \zeta^{-n}e^{nt}\prod_{j=1}^{k}(1-\zeta^{j}e^{-jt})^{-1}. \tag{3.1}$$

Then
$$P(n,k) = \sum_{q=1}^{k}\sum_{\zeta}P_\zeta(n,k) = P_1(n,k) + \sum_{q=2}^{k}\sum_{\zeta}P_\zeta(n,k), \tag{3.2}$$

where the summation is to be taken for every primitive qth root of unity ($q \leqslant k$). This remarkable formula splits $P(n,k)$ into two parts: a principal term $P_1(n,k)$ which governs the asymptotic behaviour of $P(n,k)$, and a circulatory part $\sum_{q,\zeta}' P_\zeta(n,k)$ which depends on the arithmetical properties of n.

It may seem surprising that the Sylvester formula has never been used for the purposes of an asymptotic evaluation. Hardy and Ramanujan, in their famous paper on partitions,† came to a rather unfavourable conclusion with regard to this possibility. I quote from their footnote on p. 76: 'These (cf. Sylvester's) formulae do, of course, furnish incidentally asymptotic formulae for the functions in question. But they are, from this point of view, of a very trivial character: the interest which they possess is algebraical.' The asymptotic formulae to which Hardy and Ramanujan refer here are those which hold for fixed k when n tends to infinity: $p(n,k) \cong n^{k-1}/k!\,(k-1)!$, which incidentally follows easily from the elementary formula (1.3). But there is really no need to confine ourselves to that trivial situation (when k is fixed), since Sylvester's theorem expresses an identity which holds for every n and k. As a matter of fact, the Hardy–Ramanujan formula itself is composed of expressions which strongly suggest a close relationship to the corresponding Sylvester waves for $P(n,n)$.

* See e.g. L. E. Dickson, *History of the theory of numbers* (Washington, 1920), 2, 119.

† G. H. Hardy and S. Ramanujan, *Proc. London Math. Soc.* (2) 17 (1918), 75–115.

The other theorem on which my proof is based is the following asymptotic expansion which I obtained quite recently* for the coefficients of certain power series.

THEOREM 2. *Let* $$f(z) = \sum_{\nu=1}^{\infty} \frac{1}{\nu} c_\nu z^\nu$$

be a given power series with positive radius of convergence and constant term zero, h an integer, α a complex parameter, σ, τ real constants $(0 < \sigma \leqslant 1)$. Write

$$G(z) = e^{K\{z-f(\alpha z)\}} = \sum_{\mu=0}^{\infty} \frac{1}{\mu!} A_\mu(h,\alpha) z^\mu, \qquad K = \frac{1}{\sigma}(h+\tau). \tag{3.3}$$

Let $u = u(t) = u(\sigma, t)$ denote the inverse function of

$$t = \frac{u}{\sigma + uf'(u)}, \qquad u = t\Big(\sigma + \sum_{\nu=1}^{\infty} c_\nu u^\nu\Big), \tag{3.4}$$

and let $v = v(\sigma, t)$ be defined by

$$v = t\Big(\sigma + \sum_{\nu=1}^{\infty} |c_\nu| v^\nu\Big).$$

Then

$$\log A_{h-1} = \log A_{h-1}(h,\alpha) = (h-1)\log K - K\int_0^\alpha \frac{u(t)-\sigma t}{t^2}\,dt + \tau \log\frac{u(\alpha)}{\sigma\alpha} +$$

$$+\tfrac{1}{2}\log\frac{u'(\alpha)}{\sigma} + \sum_{\mu=1}^{m-1} h^{-\mu}\phi_\mu(\alpha) + O(h^{-m}) \tag{3.5}$$

for every $m > 0$ and certain functions $\phi_\mu(\alpha)$, analytic for $|\alpha| < \rho$, where ρ denotes the radius of convergence of $v(t)$. The expansion (3.5) *is uniformly valid for $|\alpha| \leqslant \rho_0 < \rho$.*

THEOREM 3. *Suppose that the coefficients of $f(z)$ are not constants but have an asymptotic expansion of the form*

$$c_\nu = C_\nu\left\{1 + \sum_{\mu=1}^{m-1} \binom{\nu+1}{2\mu} E_\mu k^{-2\mu} + \delta_\nu E_m \binom{\nu+1}{2m} k^{-2m}\Big(1+\frac{a}{k}\Big)^\nu\right\} \tag{3.6}$$

where C_ν, E_μ, a $(\geqslant 0)$ are constants (independent of k) and $|\delta_\nu| < 1$. Then Theorem 2 still holds if we form $u(t)$ with coefficients C_ν independent of k viz.

$$u = t\Big(\sigma + \sum_{\nu=1}^{\infty} C_\nu u^\nu\Big).$$

* G. Szekeres, 'The asymptotic behaviour of the coefficients of certain power series', *Acta Sci. Math. Szeged*, 12 (1950), 187–98, Theorems 1, 2.

Corollary. *If the coefficients of $f(z)$ have the form*

$$c_\nu = C_\nu\{1+\delta_\nu(\nu+1)K^{-1}\} \quad (|\delta_\nu| < 1,\ C_\nu \textit{ constant}),$$

and

$$u = t\Big(\sigma + \sum_{\nu=1}^{\infty} C_\nu u^\nu\Big),$$

then

$$\log A_{h-1} = (h-1)\log K - K\int_0^\alpha \frac{u(t)-\sigma t}{t^2}\,dt + O(1). \tag{3.7}$$

I shall obtain the proof of Theorem 1 by applying the asymptotic expansions of Theorems 2, 3 to the coefficients of $f(\zeta,t)$ in (3.1). Write

$$R_\zeta(N,k) = P_\zeta\{N-\tfrac{1}{2}k(k+1),k\},$$

so that $R_\zeta(N,k)$ is the coefficient of t^{-1} in

$$f(\zeta,t) = \zeta^{-N+\frac{1}{2}k(k+1)}e^{Nt}\prod_{j=1}^{k}(e^{\frac{1}{2}jt}-\zeta^j e^{-\frac{1}{2}jt})^{-1}. \tag{3.8}$$

Sylvester's theorem states that

$$R(N,k) = R_1(N,k)+\sum_{q=2}^{k}\sum_{\zeta} R_\zeta(N,k). \tag{3.9}$$

4. The principal term $R_1(N,k)$

I shall prove in this section that $R_1(N,k)$ satisfies the asymptotic expansion (1.6), (1.6′). From (3.8),

$$f(1,t) = \frac{1}{k!}e^{Nt}t^{-k}\prod_{\nu=1}^{k}\tfrac{1}{2}\nu t(\sinh\tfrac{1}{2}\nu t)^{-1}. \tag{4.1}$$

Hence, writing

$$g(1,t) = \exp\Big[Nt-\sum_{\nu=1}^{k}\log\{(\tfrac{1}{2}\nu t)^{-1}\sinh\tfrac{1}{2}\nu t\}\Big] = \sum_{\nu=0}^{\infty}\frac{1}{\nu!}b_\nu(N,k)t^\nu, \tag{4.2}$$

we have

$$R_1(N,k) = \{k!(k-1)!\}^{-1}b_{k-1}(N,k). \tag{4.3}$$

From (4.2),

$$g(1,t) = \exp\Big\{Nt-\sum_{\nu=1}^{k}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{B_p}{2p(2p!)}(\nu t)^{2p}\Big\} \tag{4.4}$$

$$= \exp\Big\{Nt-\sum_{p=1}^{\infty}\Big(\sum_{\nu=1}^{k}\nu^{2p}\Big)(-1)^{p-1}\frac{B_p}{2p(2p)!}t^{2p}\Big\}$$

where B_p denotes the pth Bernoulli number ($B_0 = 1, B_1 = \frac{1}{6}, B_2 = \frac{1}{30},\ldots$). To evaluate $\sum_{\nu=1}^{k}\nu^{2p}$, we use a modified form of the Euler–Bernoulli formula. Let the numbers D_{2s} be defined by

$$D_0 = 1, \qquad \sum_{\mu=0}^{s}\binom{2s+1}{2\mu}D_{2\mu} = 0 \quad (s = 1, 2,\ldots). \tag{4.5}$$

[See Nörlund,* where it is shown that $D_{2s} = (-1)^s(2^{2s}-2)B_s$.]

* N. E. Nörlund, *Differenzenrechnung* (Berlin, 1924), 27–8.

Then, for $m > 0$, $p > 0$,

$$(2p+1)\sum_{\nu=1}^{k}\nu^{2p}$$
$$=\sum_{\mu=0}^{m-1}\binom{2p+1}{2\mu}2^{-2\mu}D_{2\mu}(k+\tfrac{1}{2})^{2p+1-2\mu}+\delta B_m\binom{2p+1}{2m}(k+1)^{2p-2m+1}$$
$$(|\delta|<1). \quad (4.6)$$

The formula is a direct consequence of a formula given by Nörlund [loc. cit. 31].

Combining (4.6) and (4.4), we obtain

$$g(1,t)=\exp\Big\{Nt-$$
$$-\sum_{\mu=0}^{m-1}2^{-2\mu}D_{2\mu}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{B_p}{2p(2p+1)!}\binom{2p+1}{2\mu}(k+\tfrac{1}{2})^{2p-2\mu+1}t^{2p}-$$
$$-B_m\sum_{p=1}^{\infty}(-1)^{p-1}\frac{B_p}{2p(2p+1)!}\binom{2p+1}{2m}(k+1)^{2p-2m+1}t^{2p}\Big\} \quad (|\delta_p|<1)$$

or equivalently, introducing

$$z=\frac{N}{k+\frac{1}{2}}t=\frac{1}{\alpha}(k+\tfrac{1}{2})t, \qquad \alpha=(k+\tfrac{1}{2})^2/N,$$
$$G(1,z)=g\Big\{1,\frac{1}{N}(k+\tfrac{1}{2})z\Big\}=\exp\Big[(k+\tfrac{1}{2})\Big\{z-\sum_{p=1}^{\infty}\frac{c_{2p}}{2p}(\alpha z)^{2p}\Big\}\Big]$$
$$=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}A_\nu(\alpha,k)z^\nu, \quad (4.7)$$

where

$$C_{2p}=(-1)^{p-1}\frac{B_p}{(2p+1)!}\Big[\sum_{\mu=0}^{m-1}2^{-2\mu}D_{2\mu}\binom{2p+1}{2\mu}(k+\tfrac{1}{2})^{-2\mu}+$$
$$+B_m\delta_p\binom{2p+1}{2m}\Big(1+\frac{1}{2k+1}\Big)^{2p-2m+1}(k+\tfrac{1}{2})^{-2m}\Big], \quad (4.8)$$

and
$$A_\nu(\alpha,k)=b_\nu(N,k)\Big(\frac{k+\frac{1}{2}}{N}\Big)^\nu=b_\nu(N,k)\Big(\frac{\alpha}{k+\frac{1}{2}}\Big)^\nu. \quad (4.9)$$

We apply Theorem 3 to $G(1,z)$, writing in (3.6)

$$\sigma=1, \qquad \tau=\tfrac{1}{2}, \qquad h=k,$$
$$C_{2p}=(-1)^{p-1}\frac{B_p}{(2p+1)!}, \qquad C_{2p-1}=0, \qquad E_\mu=2^{-2\mu}D_{2\mu}, \qquad a=1.$$

We have

$$t = u\left\{1+\sum_{p=1}^{\infty}(-1)^{p-1}\frac{B_p}{(2p+1)!}u^{2p}\right\}^{-1} = \left(\tfrac{1}{4}+u^{-2}\int_0^u \frac{x}{e^x-1}\,dx\right)^{-1},$$

$$\int_0^\alpha \{u(t)-t\}t^{-2}\,dt = 2-2\frac{u(\alpha)}{\alpha}-\log\alpha+\log\{2\sinh\tfrac{1}{2}u(\alpha)\},$$

by (1.7). Hence Theorem 3 gives

$$\begin{aligned}\log A_{k-1}(\alpha,k) &= (k-1)\log(k+\tfrac{1}{2})-\\ &\quad -(k+\tfrac{1}{2})\left\{2-2\frac{u(\alpha)}{\alpha}-\log\alpha+\log\{2\sinh\tfrac{1}{2}u(\alpha)\}\right\}+\\ &\quad +\tfrac{1}{2}\log\frac{u(\alpha)}{\alpha}+\tfrac{1}{2}\log u'(\alpha)+\sum_{\mu=1}^{m-1}\phi_\mu(\alpha)k^{-\mu}+O(k^{-m}):\end{aligned} \quad (4.10)$$

that is, by (4.9),

$$\begin{aligned}\log b_{k-1}(N,k) &= (k-1)\log(k+\tfrac{1}{2})-(k-1)\log\alpha+\log A_{k-1}(\alpha,k)\\ &= 2(k-1)\log(k+\tfrac{1}{2})+(k+\tfrac{1}{2})\left\{2\frac{u(\alpha)}{\alpha}-\log\{2\sinh\tfrac{1}{2}u(\alpha)\}\right\}-\\ &\quad -2(k+\tfrac{1}{2})+\log\alpha+\tfrac{1}{2}\log\{uu'(\alpha)\}+\sum_{\mu=1}^{m-1}\phi_\mu(\alpha)k^{-\mu}+O(k^{-m}).\end{aligned}$$

Stirling's formula implies

$$\begin{aligned}&\log\{k!(k-1)!\}\\ &\quad = 2k\log k-2k+\log(2\pi)+\sum_{\mu=1}^{m-1}(-1)^{\mu-1}\frac{B_\mu}{\mu(2\mu-1)}k^{-2\mu+1}+O(k^{-2m+1}).\end{aligned}$$

Hence, by (4.3),

$$\begin{aligned}\log R_1(N,k) &= \log b_{k-1}(N,k)-\log\{k!(k-1)!\}\\ &= (k+\tfrac{1}{2})\left[2\frac{u(\alpha)}{\alpha}-\log\{2\sinh\tfrac{1}{2}u(\alpha)\}\right]+2k\log\left(1+\frac{1}{2k}\right)-1-2\log(k+\tfrac{1}{2})+\\ &\quad +\log\alpha-\log(2\pi)+\tfrac{1}{2}\log\{uu'(\alpha)\}+\\ &\quad +\sum_{\nu=1}^{[m/2]}(-1)^\nu\frac{B_\nu}{\nu(2\nu-1)}k^{-2\nu+1}+\sum_{\mu=1}^{m-1}\phi_\mu(\alpha)k^{-\mu}+O(k^{-m})\\ &= k\left[2\frac{u(\alpha)}{\alpha}-\log\{2\sinh\tfrac{1}{2}u(\alpha)\}\right]-\log N+\frac{u(\alpha)}{\alpha}-\tfrac{1}{2}\log\left(\frac{2}{u(\alpha)}\sinh\tfrac{1}{2}u(\alpha)\right)+\\ &\quad +\tfrac{1}{2}\log u'(\alpha)-\log(2\pi)+\sum_{\mu=1}^{m-1}k^{-\mu}\psi_\mu(\alpha)+O(k^{-m}).\end{aligned}$$

This proves the statement. The range of validity of the expression is given by the radius of convergence of $v(t)$ defined by

$$t = v\left(1+\sum_{p=1}^{\infty}\frac{B_p}{(2p+1)!}v^{2p}\right)^{-1} = v^2\left(\int_0^v (2-\tfrac{1}{2}z\cot\tfrac{1}{2}z)\,dz\right)^{-1}.$$

Since the coefficients $B_p/(2p+1)!$ are positive,

$$\left|\int_0^v (2-\tfrac{1}{2}z\cot\tfrac{1}{2}z)\,dz\right| \leqslant \int_0^r (2-\tfrac{1}{2}x\cot\tfrac{1}{2}x)\,dx$$

on the circle $|v| = r$. Hence the minimum of $|t(v)|$ on $|v| = r$ is

$$M(r) = r^2\left(\int_0^r (2-\tfrac{1}{2}\cot\tfrac{1}{2}x)\,dx\right)^{-1}.$$

It follows from the Bürmann–Lagrange theorem* that $v(t)$ is certainly convergent in a circle with radius $M(r)$. The maximum $\rho = M(R)$ of $M(r)$ is where

$$\tfrac{1}{2}R(2-\tfrac{1}{2}R\cot\tfrac{1}{2}R) = \int_0^R (2-\tfrac{1}{2}x\cot\tfrac{1}{2}x)\,dx,$$

i.e. at $R = 4{\cdot}2048....$ Hence

$$\rho = R^2\left(\int_0^R (2-\tfrac{1}{2}x\cot\tfrac{1}{2}x)\,dx\right)^{-1} = R(1-\tfrac{1}{4}R\cot\tfrac{1}{2}R)^{-1} = 2{\cdot}5984....$$

5. The circulatory part

I have to show now that the circulatory part has a smaller order of magnitude than the principal term $R_1(N,k)$. This corresponds to the fact (which, however, I shall not use) that $z = 1$ is the 'strongest' singularity of the generating function

$$\{(1-z)(1-z^2)...(1-z^k)\}^{-1} = 1+\sum_{n=1}^{\infty} P(n,k)z^n.$$

In fact, I shall prove that

$$\sum_{q=2}^{k}\sum_{\zeta} R_\zeta(N,k) = o\{k^{-m}R_1(N,k)\}$$

for every $m > 0$.

We first consider the dominating term $R_{-1}(N,k)$, which can be treated very similarly to $R_1(N,k)$. We have from (3.8)

$$f(-1,t) = (-1)^{-N+\frac{1}{4}k(k+1)}2^{-k}(h!)^{-1}e^{Nt}\prod_{\nu=1}^{h'}\{\cosh(\nu-\tfrac{1}{2})t\}^{-1}\prod_{\nu=1}^{h}\nu t(\sinh\nu t)^{-1},$$

$$h = [\tfrac{1}{2}k], \qquad h' = [\tfrac{1}{2}k+\tfrac{1}{2}],$$

and $$R_{-1}(N,k) = (-1)^{-N+\frac{1}{4}k(k+1)}2^{-k}\{h!(h-1)!\}^{-1}b_{h-1}(-1,N,k), \qquad (5.1)$$

* See Hurwitz–Courant, *Funktionentheorie*, 2. Aufl. (Berlin, 1925), 139.

where

$$g(-1,t) \equiv \sum_{\nu=0}^{\infty} \frac{1}{\nu!} b_\nu(-1,N,k)t^\nu$$

$$= \exp\left\{Nt - \sum_{\nu=1}^{h'} \log\{\cosh(\nu-\tfrac{1}{2})t\} - \sum_{\nu=1}^{h} \log\left(\frac{\sinh \nu t}{\nu t}\right)\right\}$$

$$= \exp\left\{Nt - \sum_{\nu=1}^{h'} \sum_{p=1}^{\infty} (-1)^{p-1} \frac{(2^{2p}-1)B_p}{2p(2p)!} [(2\nu-1)t]^{2p} - \right.$$

$$\left. - \sum_{\nu=1}^{h} \sum_{p=1}^{\infty} (-1)^{p-1} \frac{B_p}{2p(2p)!} (2\nu t)^{2p}\right\}$$

$$= \exp\left\{Nt - \sum_{p=1}^{\infty} \left[\sum_{\nu=1}^{h'} (2\nu-1)^{2p}\right] (-1)^{p-1} \frac{(2^{2p}-1)B_p}{2p(2p)!} t^{2p} - \right.$$

$$\left. - \sum_{p=1}^{\infty} \left[\sum_{\nu=1}^{h} (2\nu)^{2p}\right] (-1)^{p-1} \frac{B_p}{2p(2p)!} t^{2p}\right\}$$

$$= \exp\left\{Nt - \tfrac{1}{2}(k+\tfrac{1}{2}) \sum_{p=1}^{\infty} (-1)^{p-1} \frac{B_p}{2p(2p+1)!} \times \right.$$

$$\left. \times [1+\delta_p(2p+1)k^{-1}][2(k+\tfrac{1}{2})t]^{2p}\right\} \quad (|\delta_p| < 1).$$

The Euler formula is used here in the form

$$\sum_{\nu=1}^{[\frac{1}{2}k+\frac{1}{2}]} (\nu-\tfrac{1}{2})^{2p}, \qquad \sum_{\nu=1}^{[\frac{1}{2}k]} \nu^{2p} = \frac{1}{2p+1}\{\tfrac{1}{2}(k+\tfrac{1}{2})\}^{2p+1} + \delta\{\tfrac{1}{2}(k+\tfrac{1}{2})\}^{2p} \quad (|\delta| < 1).$$

We change the variable t into

$$z = \frac{2N}{k+\frac{1}{2}} t = \frac{2k+1}{\alpha} t,$$

and apply the corollary of Theorem 3 to

$$\exp\left\{\tfrac{1}{2}(k+\tfrac{1}{2})\left(z - \sum_{p=1}^{\infty} (-1)^{p-1} \frac{B_p}{2p(2p+1)!} \{1+\delta_p(2p+1)k^{-1}\}(\alpha z)^{2p}\right)\right\}.$$

We obtain as in the previous section that

$$\log\left\{b_{h-1}(-1,N,k)\left(\frac{\alpha}{2k+1}\right)^{h-1}\right\} = (h-1)\log\{\tfrac{1}{2}(k+\tfrac{1}{2})\} - \tfrac{1}{2}(k+\tfrac{1}{2}) \times$$

$$\times \left\{2 - 2\frac{u(\alpha)}{\alpha} + \log\left(\frac{2}{\alpha} \sinh \tfrac{1}{2}u(\alpha)\right)\right\} + O(1),$$

$$\log b_{h-1}(-1,N,k) = 2(h-1)\log(k+\tfrac{1}{2}) - (h-1)\log\alpha -$$

$$-h\left\{2 - 2\frac{u(\alpha)}{\alpha} + \log\left(\frac{2}{a} \sinh \tfrac{1}{2}u(\alpha)\right)\right\} + O(1).$$

Hence, by (5.1) and Stirling's formula,

$$\log\{(-1)^{N-\frac{1}{2}k(k+1)}R_{-1}(N,k)\}$$
$$= -k\log 2-\log\{h!(h-1)!\}+\log b_{h-1}(-1,N,k)$$
$$= h\left\{2\frac{u(\alpha)}{\alpha}-\log(2\sinh\tfrac{1}{2}u(\alpha))\right\}-\log N+O(1).$$

Since

$$2\frac{u(\alpha)}{\alpha}-\log\{2\sinh\tfrac{1}{2}u(\alpha)\} > 2\frac{u(\alpha)}{\alpha}-\tfrac{1}{2}u(\alpha)$$
$$= 2u(\alpha)\left(\frac{1}{\alpha}-\frac{1}{u}\right) = \frac{2}{u}\int\limits_0^u \frac{x}{e^x-1}\,dx > 0$$

and $h \simeq \frac{1}{2}k$, it follows that $R_{-1}(N,k)$ has a smaller exponential order than $R_1(N,k)$ and that $R_{-1}(N,k) = o\{k^{-m}R_1(N,k)\}$ for every $m > 0$.

6. The circulatory part (continued)

Suppose now that $q > 2$ and write

$$h = \left[\frac{k}{q}\right], \qquad k = qh+r \quad (0 \leqslant r < q),$$

$$\pi(r,\zeta) = (1-\zeta)(1-\zeta^2)...(1-\zeta^r) \quad \text{if} \quad 0 < r < q,\ \pi(0,\zeta) = 1. \tag{6.1}$$

Then

$$\pi(q-1,\zeta) = (1-\zeta)...(1-\zeta^{q-1}) = \lim_{x\to 1}\frac{x^q-1}{x-1} = q,$$

and, from (3.8),

$$f(\zeta,t) = \zeta^{-N+\frac{1}{2}k(k+1)}\{\pi(r,\zeta)q^{2h}h!\}^{-1}t^{-h}e^{Nt}\times$$
$$\times \prod_{j=1}^{k}{}' (1-\zeta^j)(e^{\frac{1}{2}jt}-\zeta^j e^{-\frac{1}{2}jt})^{-1}\prod_{\nu=1}^{h}\tfrac{1}{2}q\nu t(\sinh\tfrac{1}{2}q\nu t)^{-1}.$$

Here Π' signifies that the product index j takes only those values which are not multiples of q. A similar remark applies to Σ' below.

Write

$$g(\zeta,t) = \sum_{\nu=0}^{\infty}\frac{1}{\nu!}b_\nu(\zeta,N,k)t^\nu = \exp\Big\{Nt-\sum_{j=1}^{k}{}'\log[(1-\zeta^j)^{-1}(e^{\frac{1}{2}jt}-\zeta^j e^{-\frac{1}{2}jt})]-$$
$$-\sum_{\nu=1}^{h}\log[(\tfrac{1}{2}q\nu t)^{-1}\sinh\tfrac{1}{2}q\nu t]\Big\}. \tag{6.2}$$

Then

$$R_\zeta(N,k) = \zeta^{-N+\frac{1}{2}k(k+1)}\{\pi(r,\zeta)q^{2h}h!(h-1)!\}^{-1}b_{h-1}(\zeta,N,k).$$

Now, if $l = [\frac{1}{6}q]$,

$$|\pi(r,\zeta)| = |(1-\zeta)...(1-\zeta^r)| \geqslant |(1-e^{2\pi i/q})(1-e^{4\pi i/q})...(1-e^{2l\pi i/q})|$$
$$\geqslant \prod_{\lambda=1}^{l}\frac{6\lambda}{q} = \left(\frac{6}{q}\right)^l l! \geqslant e^{-\frac{1}{6}q}.$$

Hence $$|R_\zeta(N,k)| \leqslant q^{-2h}\{h!(h-1)!\}^{-1}e^{\frac{1}{2}q}|b_{h-1}(\zeta,N,k)|. \tag{6.3}$$

To obtain an estimate for b_{h-1} we have to study the function in the exponent of (6.2). Let us write, for $\beta \neq 1$,

$$\phi(\beta,t) = -\log\frac{e^{\frac{1}{2}t}-\beta e^{-\frac{1}{2}t}}{1-\beta} = \sum_{\mu=1}^{\infty}\frac{1}{\mu!}B(\beta,\mu)t^{\mu}. \tag{6.4}$$

We have
$$B(\beta,1) = \frac{\beta+1}{2(\beta-1)},$$

$$B(\beta,\mu+2) = (\beta-1)^{-\mu-2}(L_{\mu 0}+L_{\mu 1}\beta+...+L_{\mu\mu}\beta^{\mu})\beta \quad (\mu \geqslant 0), \tag{6.5}$$

where the numbers $L_{\mu\nu}$ are positive integers satisfying the recursion

$$L_{\mu+1,\nu+1} = (\mu+1-\nu)L_{\mu\nu}+(\nu+2)L_{\mu,\nu+1} \quad (\mu > \nu \geqslant 0),$$
$$L_{00} = L_{\mu 0} = L_{\mu\mu} = 1. \tag{6.6}$$

For
$$\phi_t'(\beta,t) = \frac{(\beta-e^t)}{2(\beta+e^t)}$$

and
$$\phi_t^{(\mu+2)}(\beta,t) = (\beta-e^t)^{-\mu-2}\beta e^t\sum_{\nu=0}^{\mu}L_{\mu\nu}\beta^{\nu}e^{(\mu-\nu)t} \quad (\mu \geqslant 0),$$

as is easily verified by induction.

Writing $t = (\beta-1)x$ and making $\beta \to 1$ in (6.4) and (6.5), we obtain, since
$$\lim_{\beta\to 1}(1-\beta)^{-1}\{e^{\frac{1}{2}x(\beta-1)}-\beta e^{-\frac{1}{2}x(\beta-1)}\} = 1-x,$$

$$\frac{1}{(\mu+2)!}(L_{\mu 0}+L_{\mu 1}+...+L_{\mu\mu}) = \frac{1}{\mu+2},$$

i.e.
$$L_{\mu 0}+L_{\mu 1}+...+L_{\mu\mu} = (\mu+1)!.$$

Hence for an arbitrary primitive qth root of unity ζ ($q \geqslant 3$)

$$|B(\zeta^j,\mu)| \leqslant |1-\zeta^j|^{-\mu}(\mu-1)! \leqslant \left(\frac{q}{3\sqrt{3}}\right)^{\mu}(\mu-1)!. \tag{6.7}$$

Next I show that

$$\sum_{j=1}^{q-1}B(\zeta^j,2\nu-1) = 0, \qquad \sum_{j=1}^{q-1}B(\zeta^j,2\nu) = (-1)^{\nu}\frac{B_\nu}{2\nu}(q^{2\nu}-1) \quad (\nu = 1, 2,...). \tag{6.8}$$

Using the identity
$$\prod_{j=1}^{q-1}(a-\zeta^j) = \frac{a^q-1}{a-1} \quad (a \neq 1),$$

we obtain

$$\sum_{j=1}^{q-1} \log\frac{e^{\frac{1}{2}t}-\zeta^j e^{-\frac{1}{2}t}}{1-\zeta^j} = -\tfrac{1}{2}(q-1)t+\sum_{j=1}^{q-1}\log\frac{e^t-\zeta^j}{1-\zeta^j}$$

$$= -\tfrac{1}{2}(q-1)t+\log\prod_{j=1}^{q-1}\frac{e^t-\zeta^j}{1-\zeta^j} = -\tfrac{1}{2}(q-1)t+\log\frac{e^{qt}-1}{q(e^t-1)}$$

$$= \log\Big(\frac{2}{qt}\sinh\tfrac{1}{2}qt\Big)-\log\Big(\frac{2}{t}\sinh\tfrac{1}{2}t\Big) = \sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{B_\nu}{2\nu(2\nu)!}(q^{2\nu}-1)t^{2\nu}.$$

Then (6.8) follows from this and (6.4).

Finally we note the following formula:

Let $0 \leqslant j < q$, $h' = [(k-j)/q]$, *then*

$$\sum_{\nu=0}^{h'}(q\nu+j)^p = \frac{1}{q(p+1)}(k+\tfrac{1}{2})^{p+1}+\delta(k+q)^p \quad (|\delta| < 1). \tag{6.9}$$

This follows from Euler's formula or simply by comparing

$$\sum_{\nu=0}^{h'}\Big(\nu+\frac{j}{q}\Big)^p \quad \text{with} \quad \int\limits_0^{(1/q)(k+\frac{1}{2})} x^p\,dx.$$

Now we can develop the function in the exponent of (6.2) into a power series. For the first sum we have

$$-\sum_{j=1}^{k}{}' \log\{(1-\zeta^j)^{-1}(e^{\frac{1}{2}jt}-\zeta^j e^{-\frac{1}{2}jt})\}$$

$$= \sum_{j=1}^{k}{}' \phi(\zeta^j, jt) = \sum_{j=1}^{k}{}'\sum_{p=1}^{\infty}\frac{1}{p!}B(\zeta^j,p)j^pt^p, \quad \text{by (6.4)},$$

$$= \sum_{p=1}^{\infty}\frac{1}{p!}t^p\sum_{j=1}^{q-1}B(\zeta^j,p)\sum_{\nu=0}^{h'}(q\nu+j)^p$$

$$= \sum_{p=1}^{\infty}\frac{1}{p!}t^p\sum_{j=1}^{q-1}B(\zeta^j,p)\Big\{\frac{1}{q(p+1)}(k+\tfrac{1}{2})^{p+1}+\delta(k+q)^p\Big\}, \quad \text{by (6.9)},$$

$$= \frac{k+\frac{1}{2}}{q}\sum_{\nu=1}^{\infty}(-1)^\nu\frac{B_\nu}{2\nu(2\nu+1)!}(q^{2\nu}-1)\{(k+\tfrac{1}{2})t\}^{2\nu}+$$

$$+(q-1)\sum_{p=1}^{\infty}\frac{\delta_p}{p}\Big\{\frac{q}{3\sqrt{3}}(k+q)t\Big\}^p, \quad \text{by (6.7), (6.8)}.$$

For the second sum we have

$$-\sum_{\nu=1}^{h}\log\{(\tfrac{1}{2}q\nu t)^{-1}\sinh\tfrac{1}{2}q\nu t\}$$

$$=\sum_{\nu=1}^{h}\sum_{p=1}^{\infty}(-1)^p\frac{B_p}{2p(2p)!}(q\nu t)^{2p}=\sum_{p=1}^{\infty}(-1)^p\frac{B_p}{2p(2p)!}t^{2p}\sum_{\nu=1}^{h}(q\nu)^{2p}$$

$$=\frac{k+\frac{1}{2}}{q}\sum_{p=1}^{\infty}(-1)^p\frac{B_p}{2p(2p+1)!}\{(k+\tfrac{1}{2})t\}^{2p}+\sum_{p=1}^{\infty}\delta_p\frac{B_p}{2p(2p)!}t^{2p}(k+q)^{2p}$$

$$=\frac{k+\frac{1}{2}}{q}\sum_{p=1}^{\infty}(-1)^p\frac{B_p}{2p(2p+1)!}\{(k+\tfrac{1}{2})t\}^{2p}+\sum_{p=1}^{\infty}\frac{\delta_p}{p}\left\{\frac{q}{3\surd 3}(k+q)t\right\}^p$$

since
$$B_p/(2p)!<(q/3\surd 3)^{2p}.$$

Combining these results with (6.2), we have

$$g(\zeta,t)=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}b_\nu(\zeta,N,k)t^\nu$$

$$=\exp\left\{Nt-\frac{k+\frac{1}{2}}{q}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{B_p}{2p(2p+1)!}[q(k+\tfrac{1}{2})t]^{2p}+\right.$$

$$\left.+q\sum_{p=1}^{\infty}\frac{\delta_p}{p}\left(\frac{q}{3\surd 3}(k+q)t\right)^p\right\}.$$

Changing t into

$$z=\frac{Nq}{k+\frac{1}{2}}t=\frac{k+\frac{1}{2}}{\alpha}qt$$

and noting that

$$\alpha<\rho=2\cdot 5984...<\frac{3\surd 3(k+\frac{1}{2})}{k+q},^{*}\qquad \frac{\alpha(k+q)}{3\surd 3(k+\frac{1}{2})}<1,$$

we obtain

$$\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\left\{\frac{\alpha}{q(k+\frac{1}{2})}\right\}^\nu b_\nu(\zeta,N,k)z^\nu=G(q,z)\exp\left\{q\sum_{p=1}^{\infty}\frac{\delta_p}{p}\left(\frac{\alpha(k+q)}{3\surd 3(k+\frac{1}{2})}z\right)^p\right\}$$

$$=G(q,z)\exp\left\{q\sum_{p=1}^{\infty}\frac{\delta_p}{p}z^p\right\}$$

$$=G(q,z)\sum_{\mu=0}^{\infty}\delta_\mu\binom{q+\mu-1}{\mu}z^\mu,$$

* As a matter of fact, $\rho=2\cdot 5984...$ is slightly bigger than $\frac{1}{2}3\surd 3$ and the above inequality is not true if $h=1$ and q is very near to k. But, if $q\geqslant 4$, then obviously $3\surd 3$ can be replaced by $4\surd 2$ in (6.7), and the inequality becomes $\alpha<2\surd 2=2\cdot 8\ldots$.

where

$$G(q,z) = \exp\left\{\frac{1}{q}(k+\tfrac{1}{2})\left(z-\sum_{p-1}^{\infty}(-1)^{p-1}\frac{B_p}{2p(2p+1)!}(\alpha z)^{2p}\right)\right\} = \sum_{\nu=0}^{\infty}\frac{1}{\nu!}A_\nu z^\nu. \tag{6.10}$$

Hence

$$\frac{1}{(h-1)!}|b_{h-1}(\zeta,N,k)| \leqslant \left\{\frac{q}{\alpha}(k+\tfrac{1}{2})\right\}^{h-1}\sum_{\mu+\nu=h-1}\binom{q+\mu-1}{\mu}\frac{1}{\nu!}A_\nu. \tag{6.11}$$

Let us put $K = (k+\frac{1}{2})/q$ and apply Theorem 2 to the coefficients

$$A_{l-1}, \qquad l = \sigma K \quad (\sigma \leqslant 1).$$

We have, for $u = u(\sigma,t)$,

$$t = u\left\{\sigma+\sum_{p=1}^{\infty}(-1)^{p-1}\frac{B_p}{(2p+1)!}u^{2p}\right\}^{-1} = u\left(\sigma-1+\frac{1}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx\right)^{-1} \tag{6.12}$$

and, from (3.5),

$$\begin{aligned}\log A_{l-1} &= (l-1)\log K-K\int_0^\alpha\frac{u(\sigma,t)-\sigma t}{t^2}\,dt+O(1)\\ &= (\sigma K-1)\log K-K\left\{2\sigma+\sigma\log\frac{u(\sigma,\alpha)}{\sigma\alpha}-2\frac{u(\sigma,\alpha)}{\alpha}+\right.\\ &\qquad\left.+\log\left(\frac{2}{u(\sigma,\alpha)}\sinh\tfrac{1}{2}u(\sigma,\alpha)\right)\right\}+O(1),\end{aligned} \tag{6.13}$$

since, by (6.12), $\displaystyle \frac{\partial u}{\partial\alpha}\left(\frac{2}{\alpha}+\frac{1-\sigma}{u}-\tfrac{1}{2}\coth\tfrac{1}{2}u\right) = u\alpha^{-2}.$

Stirling's formula gives

$$\begin{aligned}\log\left\{\frac{1}{(l-1)!}A_{l-1}\right\} &= \log A_{l-1}-(l-\tfrac{1}{2})\log l+l+O(1)\\ &= K\left\{2\frac{u(\sigma,\alpha)}{\alpha}-\sigma-\sigma\log\frac{u(\sigma,\alpha)}{\alpha}-\log\left[\frac{2}{u(\sigma,\alpha)}\sinh\tfrac{1}{2}u(\sigma,\alpha)\right]\right\}-\\ &\qquad -\log K+\tfrac{1}{2}\log l+O(1).\end{aligned} \tag{6.14}$$

The expression in the brackets $\{\}$ has a maximum (at fixed α) if

$$\frac{\partial}{\partial\sigma}\left\{2\frac{u(\sigma,\alpha)}{\alpha}-\sigma-\sigma\log\frac{u(\sigma,\alpha)}{\alpha}-\log\left(\frac{2}{u(\sigma,\alpha)}\sinh\tfrac{1}{2}u(\sigma,\alpha)\right)\right\} = 0,$$

$$\frac{\partial u}{\partial\sigma}\left\{\frac{2}{\alpha}+\frac{1-\sigma}{u}-\tfrac{1}{2}\coth\tfrac{1}{2}u\right\} = 1+\log\frac{u}{\alpha}.$$

Here the left-hand side is 1, as is easily seen from (6.12). Hence (6.14) is maximum if $u(\sigma,\alpha) = \alpha$, i.e., by (6.12), if

$$\sigma = 1-\sum_{p=1}^{\infty}(-1)^{p-1}\frac{B_p}{(2p+1)!}\alpha^{2p} = 2-\frac{1}{\alpha}\int\limits_0^{\alpha}\tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx.$$

For this value of σ, the expression in (6.14) becomes

$$2-\sigma-\log\left(\frac{2}{u}\sinh\tfrac{1}{2}u\right) = \frac{1}{\alpha}\int\limits_0^{\alpha}\tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx-\log\left(\frac{2}{\alpha}\sinh\tfrac{1}{2}\alpha\right),$$

and

$$\log\left\{\frac{1}{(l-1)!}A_{l-1}\right\} < K\left\{\frac{1}{\alpha}\int\limits_0^{\alpha}\tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx-\log\left(\frac{2}{\alpha}\sinh\tfrac{1}{2}\alpha\right)\right\}.$$

Combining this with (6.11) and (6.3), and noting that

$$K = \frac{1}{q}(k+\tfrac{1}{2}) < h-1$$

and

$$\binom{q+\mu-1}{\mu} < \binom{q+h}{h} = \frac{(q+h)!}{q!\,h!},$$

in (6.11), we obtain

$$\begin{aligned}\log|R_\zeta(N,k)| &< -2h\log q-(h+\tfrac{1}{2})\log h+h+\tfrac{1}{6}q+(h-1)\log\left\{\frac{q}{\alpha}(k+\tfrac{1}{2})\right\}+\\ &\quad+(q+h)\log(q+h)-q\log q-h\log h+\\ &\qquad+h\left\{\frac{1}{\alpha}\int\limits_0^{\alpha}\tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx-\log\left(\frac{2}{\alpha}\sinh\tfrac{1}{2}\alpha\right)\right\}\\ &= h\left\{1+\frac{1}{\alpha}\int\limits_0^{\alpha}\tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx-\log(2\sinh\tfrac{1}{2}\alpha)\right\}-\log N+h\log(1+q/h)+\\ &\qquad+q\log(1+h/q)+\tfrac{1}{6}q+O(\log h).\end{aligned}$$

Here $\quad h\log(1+q/h) = O(k^{\frac{1}{2}}), \qquad q\log(1+h/q) = O(k^{\frac{1}{2}})$

since
$$\left[\frac{k}{gh}\right] = 1.$$

Hence

$$\begin{aligned}\log|R_\zeta(N,k)| &< h\left\{1+\frac{1}{\alpha}\int\limits_0^{\alpha}\tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx-\log(2\sinh\tfrac{1}{2}\alpha)\right\}-\\ &\qquad-\log N+\tfrac{1}{6}q+O(k^{\frac{1}{2}})\\ &\leqslant \tfrac{1}{2}k\left\{1+\frac{1}{\alpha}\int\limits_0^{\alpha}\tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx-\log(2\sinh\tfrac{1}{2}\alpha)+\tfrac{1}{2}\right\}-\\ &\qquad-\log N+O(k^{\frac{1}{2}}).\end{aligned}$$

In § 4 we have proved that

$$\log R_1(N,k) = k\left\{2\frac{u(\alpha)}{\alpha} - \log[2\sinh\tfrac{1}{2}u(\alpha)]\right\} - \log N + O(1).$$

Hence we have finished the proof if we can show that

$$\frac{3}{2}+\frac{1}{\alpha}\int_0^\alpha \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx - \log(2\sinh\tfrac{1}{2}\alpha) < 3\left\{2\frac{u(\alpha)}{\alpha} - \log[2\sinh\tfrac{1}{2}u(\alpha)]\right\}$$

$$= 3\left\{\frac{2}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx - \log[2\sinh\tfrac{1}{2}u(\alpha)]\right\}. \quad (6.15)$$

Now, if $u > 0$, then

(i) $\dfrac{1}{u}\displaystyle\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx$ is monotone increasing,

(ii) $\dfrac{1}{u}\displaystyle\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx - \log\left(\frac{2}{u}\sinh\tfrac{1}{2}u\right)$ is monotone decreasing,

(iii) $\dfrac{2}{u}\displaystyle\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx - \log\left(\frac{2}{u}\sinh\tfrac{1}{2}u\right)$ is monotone increasing with u.

For,

(i) $$\frac{d}{du}\left(\frac{1}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx\right) = \frac{1}{u}\left(\tfrac{1}{2}u\coth\tfrac{1}{2}u - \frac{1}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx\right) > 0$$

if $u > 0$,

(ii) $$\frac{d}{du}\left\{\frac{1}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx - \log\left(\frac{2}{u}\sinh\tfrac{1}{2}u\right)\right\}$$

$$= \frac{1}{u}\left(1 - \frac{1}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx\right) < 0 \quad \text{if } u > 0,$$

(iii) $$\frac{d}{du}\left\{\frac{2}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx - \log\left(\frac{2}{u}\sinh\tfrac{1}{2}u\right)\right\}$$

$$= \frac{1}{u^2}\left(\tfrac{1}{2}u^2\coth\tfrac{1}{2}u + u - 2\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx\right)$$

and the last expression is positive for $u > 0$, since

$$\frac{d}{du}\left(\tfrac{1}{2}u^2\coth\tfrac{1}{2}u+u-2\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx\right) = 1-\tfrac{1}{4}u^2(\operatorname{cosech}\tfrac{1}{2}u)^2 > 0.$$

Hence

$$\frac{3}{2}+\frac{1}{\alpha}\int_0^\alpha \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx-\log\left(\frac{2}{\alpha}\sinh\tfrac{1}{2}\alpha\right) \leqslant \frac{5}{2} \quad \text{by (ii)},$$

$$3\left\{\frac{2}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx-\log\left(\frac{2}{u}\sinh\tfrac{1}{2}u\right)\right\} \geqslant 6 \quad \text{by (iii)},$$

and (6.15) is true if

$$\tfrac{5}{2}-\log\alpha < 6-3\log u(\alpha), \qquad 3\log u(\alpha)-\log\alpha < \tfrac{7}{2}.$$

Now
$$\frac{u(\alpha)}{\alpha} = \frac{1}{u}\int_0^u \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx$$

is increasing with u (and α) by (i). Hence $3\log u(\alpha)-\log\alpha$ is monotone increasing with α. But, if $u(\alpha) = 3{\cdot}5$, then

$$\alpha = 3{\cdot}5^2\left(\int_0^{3\cdot 5} \tfrac{1}{2}x\coth\tfrac{1}{2}x\,dx\right)^{-1} = 2{\cdot}68... > \rho,$$

$$3\log u(\alpha)-\log\alpha < 3\log 3{\cdot}5-\log 2{\cdot}68 < \tfrac{7}{2},$$

which proves (6.15).

[*Note added 14 July 1950.*]

From this point of view it is interesting to compare by a numerical example the Sylvester terms for $P(n, n)$ and the Hardy–Ramanujan terms for $P(n)$. For $q \leqslant n$, write $P_q(n) = \sum_\zeta P_\zeta(n, n)$, where ζ runs through the primitive qth roots of unity. Taking $n = 8$ we obtain

$$P_1(8) = 21{\cdot}4127,\quad P_2(8) = 0{\cdot}4112,\quad P_3(8) = -0{\cdot}0566,\quad P_4(8) = 0{\cdot}1016,$$
$$P_5(8) = 0,\quad P_6(8) = 0{\cdot}0278,\quad P_7(8) = 0{\cdot}0408,\quad P_8(8) = 0{\cdot}0625,$$

whereas the first five terms of the modified exact Hardy–Ramanujan–Rademacher series are

$$21{\cdot}7092+0{\cdot}3463-0{\cdot}0896+0{\cdot}0500-0{\cdot}0192+\dots.$$

Of course, the sum of both series is $P(8) = 22$.

For large values of n, I suspect a very close approximation of the $P_q(n)$ by the corresponding Hardy–Ramanujan–Rademacher terms. Unfortunately, it is not easy to obtain data for comparison since the computation of the Sylvester waves becomes almost prohibitive for large values of n. In one instance I have calculated $P_2(40)$ and found $+7{\cdot}4784$, whereas the second Rademacher term for $P(40)$ is $+7{\cdot}4311$. In the computation I was kindly assisted by the Mathematical Statistics section of the Commonwealth Scientific and Industrial Research Organization.