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Asymptotic enumeration of sparse graphs with a minimum degree constraint

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Abstract

We derive an asymptotic formula for the number of graphs with n vertices all of degree at least k, and m edges, with k fixed. This is done by summing the asymptotic formula for the number of graphs with a given degree sequence, all degrees at least k. This approach requires analysis of a set of independent truncated Poisson variables, which approximate the degree sequence of a random graph chosen uniformly at random among all graphs with n vertices, m edges, and a minimum degree at least k. Our main result generalizes a result of Bender, Canfield and McKay and of Korshunov, who treated the case $k = 1$ using different methods.

1. Introduction

It is a quite fundamental question to ask for the number of graphs with n vertices, all of degree at least k. We call such a graph a *k-core*. This is only a slight abuse of the usual convention, in which a k-core is defined for a particular graph as the maximal subgraph which has minimum degree at least k. For the purposes which motivate us, we require an asymptotic formula for the number $C_k(n,m)$ of k-cores with n vertices and m edges, with k fixed. The only interesting range of m is $O(n \log n)$, since for larger m it is well known that the proportion of graphs with any vertices of degree less than k is exceedingly small.

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We do not consider the trivial case $k = 0$. The 1-cores are precisely the graphs with no isolated vertices. For this case, such asymptotic formulae were found independently by Korshunov [8] and Bender et al. [5], the latter obtaining bounds on the remainder term. The approach in [5] was based on a recurrence equation for the number of such graphs. Extending this appealing idea to $k > 1$ is quite problematic. There were several approaches in [8], one of which studied this problem by considering the distribution of the number of isolated vertices in a random graph with n vertices and m edges. Another, which applied for $m < n/2 + n^{2/3}/\log n$, was to show that such a graph is with high probability a forest with maximum tree size at most 4. Again, these methods look very difficult to extend beyond $k = 1$.

In [13, Proposition 2] (see also [14, Proof of Theorem 3.1]), an entirely different method was used which is much simpler to implement than either of these, and the results apply to the more general problem of arbitrary k. This method is to sum the well known asymptotic formula for the number of graphs with given degree sequence, over the appropriate degree sequences. The result in [13] permits an additional number of vertices to have specified degrees less than k. However, it does not cover the cases that $m/n \to \infty$ or $2m - kn = o(n)$. The latter is more delicate computationally than $2m - kn = cn$, but is especially interesting for transitional effects, since when $2m - kn = 0$ the graphs are k-regular. Our aim here is to use this method to give a formula for the complete interesting range of m.

Our main result is Theorem 2, the desired asymptotic formula for the number of k-cores with a given number of vertices and edges, stated below. After this, we make some observations which indicate the flavor of our proof, and provide some upper bounds on the numbers which are useful in work to appear later. Our later work will include asymptotic enumeration of 2-connected graphs by vertices and edges (particularly in the interesting case when the graphs are quite sparse). It will also include (again in the sparse case) results on the distribution of random variables relating to the 2-cores of random graphs, properties of random connected graphs (such as the distribution of short cycles), and a simpler derivation of the asymptotic formula given by Bender et al. [4] for the number of connected graphs with n vertices and m edges. The same method also forms the basis for results on directed graphs.

Łuczak [9] showed that a random graph with given degree sequence, with all degrees between 3 and $n^{0.02}$, has connectivity equal to its minimum degree with probability asymptotic to 1. It follows that for $k \geqslant 3$, a random k-core with n vertices and m edges is k-connected with probability tending to 1 as $n \to \infty$. Hence, our main result gives, for $k \geqslant 3$, an asymptotic formula for the number of k-connected graphs with a given number of vertices and edges. (We have a cutoff $m = O(n \log n)$, but it is well known that above this range, almost all graphs are k-connected; this was shown by Erdős and Rényi [7].)

For a sequence $\vec{d} = (d_1, \ldots, d_n)$, put $2m = \sum_{j=1}^{n} d_j$ and $d_{\max} = \max_i\{d_i\}$. Let $G(\vec{d})$ be the set of graphs with degree sequence $\vec{d}$ (which is only nonempty if m is an integer), and put $g(\vec{d}) = |G(\vec{d})|$. The formula we require is the following, shown first by Bender and Canfield [3] for $d_{\max}$ bounded, and later by McKay [10] in the generality we require.

Theorem 1. *Let $\vec{d}$ be a function of n such that $m = m(n) \to \infty$ and $d_{\max} = o(m^{1/4})$ as $n \to \infty$, and m is an integer for all n. Then*

$$g(\vec{d}) = \frac{(2m-1)!!}{\prod_{j=1}^{n} d_j!} \exp\left(-\frac{\eta(\vec{d})}{2} - \frac{\eta^2(\vec{d})}{4} + O\left(\frac{d_{\max}^4}{m}\right)\right), \tag{1}$$

where

$$\eta(\vec{d}) := \frac{1}{2m} \sum_{j=1}^{n} d_j(d_j - 1). \tag{2}$$

In this theorem, as with all our asymptotic statements in which the setting is not explicitly stated, we follow the convention that the implicit error function is uniform over all possibilities for $\vec{d}$, and any variables defined directly from it, subject to whatever constraints have been explicitly imposed to be in force at the time, provided $n \to \infty$.

In fact, there is a sharper formula by McKay and Wormald [11], under the weaker condition $d_{\max} = o(m^{1/3})$. The proofs use the pairing model, which is a probabilistic space valid for any nonnegative integer sequence $\vec{d}$ with even sum. (See [6] or [15] for more details.) The basic element is a random pairing, and, by the multivariate analogue of [15, equation (2)],

$$U(\vec{d}) = \frac{g(\vec{d}) \prod_{j=1}^{n} d_j!}{(2m-1)!!} \tag{3}$$

is the probability that this random pairing corresponds to a simple graph. As it is a probability, we may immediately conclude the useful bound

$$g(\vec{d}) \leqslant \frac{(2m-1)!!}{\prod_{j=1}^{n} d_j!} \tag{4}$$

for all $\vec{d}$. For $d_{\max} = o(m^{1/4})$, $U(\vec{d})$ is evaluated asymptotically by the exponential factor in (1).

Stating our main result requires some preliminaries. We begin by introducing a family of random variables basic for this work. Denote by $Y = Y(k, \lambda)$ a random variable which has a k-truncated Poisson distribution with a parameter λ, that is

$$\mathbf{P}(Y = j) = \mathbf{P}(Y(k,\lambda) = j) = \begin{cases} \dfrac{\lambda^j}{j! f_k(\lambda)}, & j \geqslant k, \\ 0, & j < k, \end{cases} \tag{5}$$

where

$$f_a(\lambda) = e^{\lambda} - \sum_{i=0}^{a-1} \frac{\lambda^i}{i!} = \sum_{i \geqslant a} \frac{\lambda^i}{i!}.$$

In particular, $f_0(\lambda) = e^{\lambda}$, and we will find it convenient to define $f_\ell(\lambda) = e^{\lambda}$ for all $\ell \leqslant 0$. Let

$$c = \frac{2m}{n}$$

and let λ_c denote the root of the equation

$$\frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} = c, \tag{6}$$

or equivalently

$$\mathbf{E}Y = c. \tag{7}$$

It is easily seen that λ_c minimizes $f_k(\lambda)^n/\lambda^{2m}$. It follows (see the comments after the statement of Theorem 2) that λ_c maximizes $\mathbf{P}(\sum_{j=1}^n Y_j = 2m)$ as well, where $Y_1, \ldots, Y_n$ are independent copies of Y. Also define

$$\bar{\eta}_c = \lambda_c f_{k-2}(\lambda_c)/f_{k-1}(\lambda_c), \tag{8}$$

and for convenience define

$$r = (c-k)n = 2m - kn.$$

These definitions apply throughout this paper.

We now state our asymptotic formula for the number of k-cores. For the statement of this theorem, write $\lambda = \lambda_c$ and $\bar{\eta} = \bar{\eta}_c$.

Theorem 2. *Let $k \geqslant 1$ be fixed. Suppose $n, m \to \infty$ in such a way that $r \geqslant 0$ and $m = O(n \log n)$.*

(a) *If $r \to \infty$:*

$$C_k(n,m) = (1 + O(r^{-1} + r^{1/2}n^{-1+\varepsilon}))\frac{(2m-1)!!f_k(\lambda)^n}{\lambda^{2m}e^{\bar{\eta}/2+\bar{\eta}^2/4}\sqrt{2\pi nc(1+\bar{\eta}-c)}}$$

for any $\varepsilon > 0$;

(b) *for $r = O(n^{2/5})$:*

$$C_k(n,m) = (1 + O(r^{5/2}n^{-1} + \beta))\frac{(2m-1)!!f_k(\lambda)^n r^r}{\lambda^{2m}e^{\bar{\eta}/2+\bar{\eta}^2/4+r}r!}$$

where, for any $\varepsilon > 0$,

$$\beta = \begin{cases} \min\{e^{-r^{\varepsilon}}, n^{-1/2+\varepsilon}\}, & k = 1, \\ \min\{e^{-r^{\varepsilon}}, r^{1/2}n^{-2/3}\}, & k \geqslant 2. \end{cases}$$

Note 1. The factor $(2m-1)!!$ can be replaced by $\sqrt{2}(2m/e)^m$ since the error $O(m^{-1})$ in Stirling's formula for $m!$ is subsumed by the other error terms.

Note 2. We can compare to the result in [5] for the case $k = 1$, which was treated there. For this case, the leading term of our estimates agrees with that in [5], as our λ_c

is equal to $2xy$ in [5]. Our error bound improves that in [5] for all $n^{1/2+\varepsilon} < r = O(n \log n)$. It is suggested in [5] that the true correction term to the leading term in the form given in [5] ($k = 1$) is actually $O(1/m)$. Not contradicting this, we believe that the error bound for our result in (a) cannot be lower than $O(r^{-1})$ for any k (see (22)).

To explain the appearance of truncated Poisson variables in presenting the formulae, note that the right-hand side of the bound (4) (see (1) too) leads us to consider

$$Q_k(n,m) = \sum_{\substack{d_1,\dots,d_n \geqslant k \\ d_1+\cdots+d_n=2m}} \prod_{j=1}^{n} \frac{1}{d_j!}. \tag{9}$$

With $Y_1, \dots, Y_n$ defined as independent copies of $Y(k,\lambda)$ as in (5),

$$\begin{aligned}\mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right) &= \sum_{\substack{d_1,\dots,d_n \geqslant k \\ d_1+\cdots+d_n=2m}} \prod_{j=1}^{n} \frac{\lambda^{d_j}}{d_j! f_k(\lambda)} = \frac{\lambda^{2m}}{f_k(\lambda)^n} \\ &\quad \times \sum_{\substack{d_1,\dots,d_n \geqslant k \\ d_1+\cdots+d_n=2m}} \prod_{j=1}^{n} \frac{1}{d_j!}\end{aligned}$$

and thus

$$Q_k(n,m) = \frac{f(\lambda)^n}{\lambda^{2m}} \mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right). \tag{10}$$

We may take this as an alternative definition of $Q_k(n,m)$. Although it is in terms of the local probability in the sum of independent copies of $Y(k,\lambda)$, by its original definition (9), it does not depend on λ. Since by (4), $(2m-1)!!\, Q_k(n,m)$ is an upper bound for $\sum_{\vec{d}} g(\vec{d})$, it is natural to choose $\lambda = \lambda_c$, the minimum point of this fraction. Of course, that same λ_c must be the maximum point for $\mathbf{P}(\sum_j Y_j = 2m)$. The fact that for the minimum point λ_c we must have $\mathbf{E}(Y(k,\lambda_c)) = 2m/n$, as determined by (6) and (7), makes this "coincidence" even less mysterious.

The next result is a more precise version of Theorem 2.

Theorem 3. *Let $\varepsilon > 0$ and $k \geqslant 0$ be fixed. For any $r \geqslant 0$,*

$$C_k(n,m) = (1 + O(\xi)) \frac{(2m-1)!!\, Q_k(n,m)}{e^{\bar{\eta}/2+\bar{\eta}^2/4}}, \tag{11}$$

where

$$\xi = \begin{cases} \min\{e^{-r^{\varepsilon}} + r^{1/2} n^{-1+\varepsilon}, n^{-1/2+\varepsilon}\}, & k \leqslant 1, \\ \min\{e^{-r^{\varepsilon}} + r^{1/2} n^{-1+\varepsilon}, r^{1/2} n^{-2/3}\}, & k \geqslant 2. \end{cases}$$

Note. For $Q_k(n,m)$ one can use (10) with $\lambda = \lambda_c$. Then the local probability in (10) is estimated in Theorem 4(a).

Theorem 3 enables one to estimate $C_k(n,m)/C_k(n',m')$, with n' and m' close to n and m, with high accuracy. Using (11), for both numerator and denominator, leads to estimating the ratio of local probabilities, which can lead to a considerably more accurate result than by use of Theorem 2.

Some preliminary investigation will reveal the relevance of a conditional expectation examined in the next section. With x as a formal variable, (3) gives

$$\sum_{2m \geqslant kn} x^{2m} \sum_{\substack{d_1,\dots,d_n \geqslant k \\ d_1+\cdots+d_n=2m}} \frac{g(\vec{d})}{(2m-1)!!} = \sum_{d_1,\dots,d_n \geqslant k} U(\vec{d}) \prod_{j=1}^{n} \frac{x^{d_j}}{d_j!},$$

so that

$$\begin{aligned} C_k(n,m) &= \sum_{\substack{d_1,\dots,d_n \geqslant k \\ d_1+\cdots+d_n=2m}} g(\vec{d}) \\ &= (2m-1)!! \cdot [x^{2m}] \sum_{d_1,\dots,d_n \geqslant k} U(\vec{d}) \prod_{j=1}^{n} \frac{x^{d_j}}{d_j!}. \end{aligned} \tag{12}$$

Picking $\lambda > 0$, the probability generating function of $Y(k,\lambda)$ is

$$\mathbf{E}(x^{Y(k,\lambda)}) = \frac{1}{f_k(\lambda)} \sum_{d \geqslant k} \frac{x^d \lambda^d}{d!} = \frac{f_k(\lambda x)}{f_k(\lambda)}.$$

So, considering the independent copies $Y_1, \dots, Y_n$ of $Y(k,\lambda)$, (12) is ("magically") transformed into

$$\begin{aligned} C_k(n,m) &= (2m-1)!! \frac{f_k(\lambda)^n}{\lambda^{2m}} \cdot [x^{2m}] \sum_{d_1,\dots,d_n \geqslant k} U(\vec{d}) \cdot \prod_{j=1}^{n} \frac{(\lambda x)^{d_j}/d_j!}{f_k(\lambda)} \\ &= (2m-1)!! \frac{f_k(\lambda)^n}{\lambda^{2m}} \mathbf{E}\left(U(\vec{Y}) \mathbf{I}_{\left\{\sum_j Y_j = 2m\right\}} \right) \\ &= (2m-1)!! \frac{f_k(\lambda)^n}{\lambda^{2m}} \mathbf{E}\left(U(\vec{Y}) \,\middle|\, \sum_{j=1}^{n} Y_j = 2m \right) \mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m \right) \end{aligned}$$

$$C_k(n,m) = (2m-1)!!\, Q_k(n,m)\, \mathbf{E}\left(U(\vec{Y}) \,\middle|\, \sum_{j=1}^{n} Y_j = 2m \right) \tag{13}$$

by (10). The last factor denotes the expected value of $U(\vec{Y})$, conditional on the indicated event. This is studied in Theorem 4.

We close this section with two upper bounds. First, combining (10) with (4) gives an upper (Chernoff-type) bound

$$C_k(n,m) \leqslant (2m-1)!! \frac{f_k(\lambda)^n}{\lambda^{2m}}, \quad \forall \lambda > 0. \tag{14}$$

Of course, to get the most out of this bound one would want to use $\lambda = \lambda_c$, since this is the minimum point of the function in question. Comparing this bound with (11), we see that the main difference is absence of the square root factors in (14). Their total product is of order $\sqrt{n\lambda_c}$. With a bit of extra work, based on the Cauchy integral formula and an inequality

$$|f_k(z)| \leqslant f_k(|z|) \exp\left(-\frac{|z| - \operatorname{Re} z}{k+1}\right),$$

(see [12]), the bound (14) can be improved to

$$C_k(n,m) \leqslant a(2m-1)!! \frac{f_k(\lambda)^n}{\lambda^{2m}\sqrt{n\lambda}}, \quad \forall \lambda > 0, \tag{15}$$

where a is an absolute constant.

The next section gives the required properties of the joint distribution of the Y_j, and the third section then proves Theorem 2.

2. Properties of truncated Poisson variables

For later use we compute here

$$\mathbf{E}(Y(Y-1)) = \frac{1}{f_k(\lambda_c)} \sum_{j \geqslant k} j(j-1) \frac{\lambda_c^j}{j!} = \frac{\lambda_c^2 f_{k-2}(\lambda_c)}{f_k(\lambda_c)} = c\bar{\eta}_c \tag{16}$$

and, using (6), (7),

$$\begin{aligned} \mathbf{Var}(Y) &= \mathbf{E}[(Y)_2] + \mathbf{E}(Y) - \mathbf{E}^2(Y) \\ &= \frac{\lambda_c^2 f_{k-2}(\lambda_c)}{f_k(\lambda_c)} + \frac{\lambda_c f_{k-1}(\lambda_c)}{f_k(\lambda_c)} - \left(\frac{\lambda_c f_{k-1}(\lambda_c)}{f_k(\lambda_c)}\right)^2 \qquad (17) \\ &= c(1 + \bar{\eta}_c - c). \qquad (18) \end{aligned}$$

Lemma 1. *The root λ_c of* (6) *exists uniquely, and*

(a) *if $2m/n \to k$ then $\lambda_c = (k+1)(c-k) + O((c-k)^2)$,*
(b) *$\lambda_c \leqslant 2m/n$ always,*
(c) *if $m/n \to \infty$ then $\lambda_c \sim 2m/n$.*

Proof. We first observe that $\mathbf{E}Y(k,\lambda)$ is monotonically increasing in λ. Perhaps the simplest way to see this is to note that, by (6), (17), and $f'_\ell(\lambda) = f_{\ell-1}(\lambda)$,

$$\frac{d\mathbf{E}(Y(k,\lambda))}{d\lambda} = \frac{d}{d\lambda}\frac{\lambda f_{k-1}(\lambda)}{f_k(\lambda)} = \frac{1}{\lambda}\mathbf{Var}(Y(k,\lambda)) > 0,$$

a relation used substantially by Pittel et al. [13] and Aronson et al. [1].

Note that for $\lambda \to 0$,

$$\lambda f_{k-1}(\lambda)/f_k(\lambda) = k + \lambda/(k+1) + O(\lambda^2) \sim k, \tag{19}$$

and for $\lambda \to \infty$, $f_{k-1}(\lambda) \sim f_k(\lambda)$. These facts together with the monotonicity mentioned above show that (6) has a unique root. Then (a) and (c) follow also from the equality in (19). Finally, from (6), $\lambda_c \leqslant c$, which gives (b). $\square$

We also note in the following lemma that $\mathbf{Var}(Y)$ (see (18) is of exact order λ, just like the usual Poisson(λ), whose variance simply equals λ.

Lemma 2. *Uniformly for all $\lambda \in (0,\infty)$,*

$$\mathbf{Var}(Y(k,\lambda_c)) = c(1 + \bar{\eta}_c - c) = \Theta(\lambda_c) = \Theta(c-k).$$

Proof. The first equality is (18). If $c \to k$ then by Lemma 1(a), $\lambda_c \to 0$. Apply (19) to (8) with k replaced by $k-1$ (unless $k=1$, in which case use $\bar{\eta}_c = \lambda_c$), to obtain

$$c(1+\bar{\eta}_c - c) = c\left(\frac{\lambda_c}{k} - (c-k) + O(\lambda_c^2)\right) \sim c(c-k)/k \sim c-k \tag{20}$$

by Lemma 1(a). The lemma follows in this case.

On the other hand, suppose that c is bounded away from k. Then by (19), λ is bounded away from 0. Lemma 1(b) and (c) then give $\lambda_c = \Theta(c-k)$. Also, since Y has the distribution of Poisson(λ) with a few values omitted, it follows that $\mathbf{Var}(Y(k,\lambda_c))$ is at least a positive constant times λ. Finally, $\mathbf{Var}(Y(k,\lambda_c)) = O(\lambda)$ since

$$\bar{\eta}_c = \frac{(k-2)\lambda_c^{k-1}/(k-1)! + cf_k(\lambda_c)}{\lambda_c^{k-1}/(k-1)! + f_k(\lambda_c)} \leqslant c, \tag{21}$$

and so the expression in (18) is $O(c) = O(\lambda)$. $\square$

We require some facts involving the event that the sum of a set of independent truncated Poisson variables has a given sum. In the rest of this section, we drop the subscripts c on λ and $\bar{\eta}$, so λ is the root of (6) and $\bar{\eta}$ is the quantity $\bar{\eta}_c$ in (8). Recall that k is fixed, and recall η defined in (2).

Theorem 4. *Let $k \geqslant 0$ be fixed. Suppose $n, m \to \infty$ in such a way that $m = O(n \log n)$. Let $Y_1, \ldots, Y_n$ be independent copies of $Y(k, \lambda)$ as in* (5). *Then*

(a) *for $r \to \infty$*

$$\mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right) = \frac{1 + O(r^{-1})}{\sqrt{2\pi nc(1 + \bar{\eta} - c)}}, \tag{22}$$

whilst for $r = O(n^{2/5})$

$$\mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right) = (1 + O(r^{5/2}n^{-1}))e^{-r}\frac{r^r}{r!}. \tag{23}$$

(b)

$$\mathbf{E}\left(e^{-\eta(\vec{Y})/2-\eta^2(\vec{Y})/4}\,\middle|\,\sum_{j=1}^{n} Y_j = 2m\right) = (1 + \tau)e^{-\bar{\eta}/2-\bar{\eta}^2/4},$$

where for all r and k and any $\varepsilon > 0$

$$\tau = O(n^{-1/2+\varepsilon}), \tag{24}$$

for $r = O(n^{1-\varepsilon})$ and any k and $\varepsilon > 0$

$$\tau = O(e^{-r^{\varepsilon}} + r^{1/2+\varepsilon}n^{-1}), \tag{25}$$

whilst for $r = o(n)$ and $k \geqslant 2$

$$\tau = O(r^{1/2}n^{-2/3}). \tag{26}$$

Note 1. The approximate size of the expression in the square root in (22) can be obtained from Lemma 2, and more precisely from (20) in the case $\lambda \to 0$.

Note 2. Estimate (22) blends with (23) since, by (20), $\mathbf{Var}(Y(k, \lambda)) \sim r/n$ for $r \to \infty$, $r = o(n)$. The domains $r \to \infty$ and $r = O(n^{2/5})$ overlap, and the approximation (23) becomes sharper than (22) once r falls below $n^{2/7}$.

Proof of Theorem 4. For (a), first let $r \to \infty$. One can easily obtain the main term in (22) (i.e. without the specific bound on the rate of convergence of the error term) as follows. The Berry-Esseen inequality establishes asymptotic normality of $\sum Y_j$, and then [2, Lemma 2] implies a local limit theorem (since the truncated Poisson distribution is log-concave, and the convolution of log-concave sequences is log-concave). The usual way to express the main term is $1/\sqrt{2\pi n \mathbf{Var}\, Y(k, \lambda)}$, which by (18) is equal to the stated term. The more precise statement in (22), with the error term, was proved in [1] for $k=2$, under the condition $n\mathbf{Var}(Y)\to\infty$, which is satisfied by Lemma 2. The argument used there extends with virtually no changes to any $k\geqslant 0$.

Suppose now that $r=O(n^{2/5})$. Consider $r>0$, as the case $r=0$ is obvious. As $r=o(n)$, we have

$$\lambda\sim\frac{(k+1)r}{n}=O(r/n),$$

see Lemma 1(a). Introducing $Y_j'=Y_j-k$, we can write

$$\mathbf{P}\left(\sum_j Y_j=2m\right)=\mathbf{P}\left(\sum_j Y_j'=r\right).$$

Now

$$\mathbf{P}(Y_j'=1)=\frac{\lambda^{k+1}/(k+1)!}{f_k(\lambda)}=\frac{\lambda}{k+1}(1+O(\lambda)),$$

and $\mathbf{P}(Y_j'\geqslant 2)=O(\lambda^2)$, so by Lemma 1(a)

$$p:=\mathbf{P}(Y_j'\geqslant 1)=r/n+O(r^2n^{-2})$$

and

$$\sum_j\mathbf{P}(Y_j'\geqslant 2)=O(n\lambda^2)=O(r^2n^{-1})\to 0.$$

Therefore, introducing $\mathcal{Y}_j'=\min\{Y_j',1\}$,

$$\mathbf{P}\left(\sum_j Y_j'\neq\sum_j\mathcal{Y}_j'\right)=O(r^2n^{-1}).$$

Consequently

$$\begin{aligned}\mathbf{P}\left(\sum_{j=1}^n Y_j=2m\right)&=O(r^2n^{-1})+\binom{n}{r}p^r(1-p)^{n-r}\\&=O(r^2n^{-1})+e^{-r}\frac{r^r}{r!}(1+O(r^2n^{-1})),\end{aligned}$$

which gives (23), as the explicit term in the last expression is of order $r^{-1/2}$.

We will have occasion to use a very rough bound on the upper tail probability for Y:

$$\mathbf{P}(Y\geqslant j_0)=\sum_{j\geqslant j_0}\frac{\lambda^j}{j!f_k(\lambda)}=O(\exp(-j_0/2))\quad\text{for } j_0>2e\lambda. \tag{27}$$

This follows because the ratio of consecutive terms is at most $1/e$ for $j>j_0/2$, and also because each term is a probability (so at most 1).

We now turn to part (b). We will show that for the purpose of estimating $\eta=\eta(\vec{Y})$ by its expected value, the concentration of its distribution is sufficiently strong to overpower conditioning on the relatively "thin" event $\{\sum_j Y_j = 2m\}$ as in (37). Set

$$S = \eta(\vec{Y})/2 = \frac{1}{4m}\sum_{j=1}^{n} Y_i(Y_i - 1). \tag{28}$$

Then, using (16),

$$\mathbf{E}S = \frac{n}{4m}E(Y(Y-1)) = \bar{\eta}/2 = O(\log n) \tag{29}$$

by (21), and therefore

$$\begin{aligned} S + S^2 - \mathbf{E}S - (\mathbf{E}S)^2 &= (S - \mathbf{E}S)^2 + (S - \mathbf{E}S)(1 + 2\mathbf{E}S) \\ &= O(|S - \mathbf{E}S|^2 + |S - \mathbf{E}S| \log n). \end{aligned} \tag{30}$$

Let $Z_i = Y_i(Y_i - 1) - \mathbf{E}(Y_i(Y_i - 1))$, and put $z = \log^6 n$. Then (for n large)

$$\mathbf{P}(|Z_i| \geqslant z) \leqslant \mathbf{P}(Y^2 \geqslant z) = \mathbf{P}(Y \geqslant \sqrt{z}) \leqslant \exp(-\Theta(\log^3 n))$$

by (27) and Lemma 1(b). Virtually the same argument, using an obvious analogue of (27), gives

$$|\mathbf{E}(Z_i\, I_{|Z_i| \geqslant z})| \leqslant \exp(-\Theta(\log^3 n)). \tag{31}$$

Now set $Z_i^* = Z_i\, I_{|Z_i| < z}$, so that $|Z_i^*| < z$. By the Azuma–Hoeffding inequality

$$\mathbf{P}\left(\left|\sum_i (Z_i^* - \mathbf{E}Z_i^*)\right| \geqslant \alpha\right) \leqslant 2\exp(-\alpha^2/8z^2 n) \tag{32}$$

for all $\alpha > 0$. Since $\mathbf{E}Z_i = 0$,

$$\left|\sum_i \mathbf{E}Z_i^*\right| = \left|-\sum_i \mathbf{E}(Z_i\, I_{|Z_i| \geqslant z})\right| \leqslant \exp(-\Theta(\log^3 n))$$

by (31). So (32) implies that for $t = n^{1/2}\log^8 n$

$$\begin{aligned} \mathbf{P}\left(\left|\sum_i Z_i\right| \geqslant t\right) \leqslant & \sum_i \mathbf{P}(|Z_i| \geqslant z) \\ & + \mathbf{P}\left(\left|\sum_i (Z_i^* - \mathbf{E}Z_i^*)\right| \geqslant t/2\right) + \sum_i |\mathbf{E}Z_i^*| \\ \leqslant & \exp(-\Theta(\log^3 n)) + 2\exp(-n(\log^4 n)/32n) \\ \leqslant & \exp(-\Theta(\log^3 n)). \end{aligned}$$

Consequently,

$$\mathbf{P}(|S - \mathbf{E}S| \geqslant n^{1/2} m^{-1} \log^8 n) \leqslant \exp(-\Theta(\log^3 n)). \tag{33}$$

Notice also that on the event $\{|S - \mathbf{E}S| \leqslant m^{-1/2}\log^8 n\}$,

$$|S + S^2 - \mathbf{E}S - (\mathbf{E}S)^2| = O(m^{-1/2}\log^9 n)$$

by (30), and the fact that $S \geqslant 0$ always. So (33) implies

$$\begin{aligned}
&\mathbf{E}\left(\exp(-S-S^2)\middle|\sum_{j=1}^{n} Y_j = 2m\right) \\
&\quad = \exp(-\mathbf{E}S - (\mathbf{E}S)^2 + O(m^{-1/2}\log^9 n)) + \frac{\exp(-\Theta(\log^3 n))}{\mathbf{P}\left(\sum_{j=1}^{n} Y_j = 2m\right)} \\
&\quad = \exp(-\bar{\eta}/2 - \bar{\eta}^2/4 + O(m^{-1/2}\log^9 n)) + \exp(-\Theta(\log^3 n))
\end{aligned} \tag{34}$$

by (22), (29) and Lemma 2. This implies (b) with ξ given in (24) since $m \geqslant n$.

For (25) we have that $r = O(n^{1-\varepsilon})$, and hence $\lambda = O(n^{-\varepsilon})$ by Lemma 1(a). Put $T = \lceil 1/\varepsilon \rceil$, so that $n^{-4T\varepsilon} = O(n^{-4})$, and put $z = (4T+k)^2$ (noting that z is now bounded). Define Z_i as above, and note that $\mathbf{P}(Y > \sqrt{z}) = O(n^{-4T\varepsilon}) = O(n^{-4})$ using (5). Hence the argument leading to (31) now produces

$$|\mathbf{E}(Z_i\, I_{|Z_i| \geqslant z})| = O(n^{-4}). \tag{35}$$

Define $Z_i^* = Z_i\, I_{|Z_i|<z}$ as before, and set

$$W_i = Z_i^* - \mathbf{E}Z_i^*.$$

For sharp concentration of the sum of W_i, we use a common approach for large deviation inequalities. In this case, W_i takes on only a finite set of values $\{u_0, \ldots, u_\ell\}$ where $u_0 = k(k-1) - \mathbf{E}(Y(Y-1)) - \mathbf{E}Z_i^*$, and $u_j - u_0$ is a positive integer less than z for all $0 < j \leqslant \ell$. Letting $p_j = \mathbf{P}(W_i = u_j)$, we have

$$p_0 = 1 - O(\lambda), \quad \text{hence} \sum_{j>0} p_j = O(\lambda), \quad \text{and } u_0 = O(\lambda). \tag{36}$$

From these equations and Taylor's theorem, it follows that for $h = o(1)$ (to be chosen shortly)

$$\mathbf{E}(e^{hW_i}) = \sum_j p_j e^{hu_j} = 1 + h\sum_j p_j u_j + \frac{1}{2}h^2 \sum_j p_j u_j^2 + O(h^3\lambda).$$

The first summation is $\mathbf{E}W_i = 0$. Letting V denote the second summation (which happens to be $\mathbf{E}W_i^2$), we have $V = \Theta(\lambda)$, and so

$$\log \mathbf{E}(e^{hW_i}) = h^2 V/2 + O(h^3\lambda).$$

Thus for any $\alpha > 0$, using Markov's inequality for the second step,

$$\begin{aligned}\mathbf{P}\left(\sum_{i=1}^{n} W_i \geqslant \alpha\right) &= \mathbf{P}\left(e^{h\sum_{i=1}^{n} W_i} \geqslant e^{h\alpha}\right)\\ &\leqslant e^{-h\alpha}\mathbf{E}\left(e^{h\sum_{i=1}^{n} W_i}\right)\\ &= e^{-h\alpha}\left(\mathbf{E}(e^{hW_i})\right)^n\\ &= \exp(-h\alpha + nh^2V/2 + O(nh^3\lambda)).\end{aligned}$$

Selecting $h = \alpha/Vn$ to minimize the quadratic, this bound becomes

$$\exp(-\alpha^2/2Vn + O(nh^3\lambda)) = \exp(-\alpha^2/2Vn + O(\alpha^3/r^2))$$

since $h = \Theta(\alpha/r)$ and $\lambda = \Theta(r/n)$. To satisfy the requirement $h = o(1)$, we shall restrict α to $o(r)$. For such α we now have

$$\mathbf{P}\left(\sum_{i=1}^{n} W_i \geqslant \alpha\right) \leqslant \exp(-\Theta(\alpha^2/r)).$$

The same argument clearly bounds $\mathbf{P}(\sum_{i=1}^{n} W_i \leqslant -\alpha)$ by an identical quantity, since it applies when all the values u_j are negated. Applying this with $\alpha = r^{1/2+\varepsilon}$ say, gives

$$\mathbf{P}\left(\left|\sum_i (Z_i^* - \mathbf{E}Z_i^*)\right| \geqslant r^{1/2+\varepsilon}\right) = O(\exp(-r^{3\varepsilon/2}))$$

for any fixed $\varepsilon > 0$. Using this in place of (32), and (35) in place of (31), the earlier argument now yields, instead of (33),

$$\mathbf{P}(|S - \mathbf{E}S| \geqslant 2m^{-1}r^{1/2+\varepsilon}) \leqslant \exp(-\Theta(\log^3 n)) + O(\exp(-r^{3\varepsilon/2})).$$

Since in this case $\mathbf{E}S = O(1)$, in place of (30) we use $S + S^2 - \mathbf{E}S - (\mathbf{E}S)^2 = O(|S - \mathbf{E}S|^2 + |S - \mathbf{E}S|)$. The right-hand side of (34) becomes

$$\exp(-\bar{\eta}/2 - \bar{\eta}^2/4 + O(m^{-1}r^{1/2+\varepsilon})) + O(\exp(-r^{\varepsilon}))$$

and we have (b) with the form of ξ in (25). Note that other bounds are obtained with different choices of α; our choice here is motivated by the type of bound which will eventuate in Theorem 2.

For (26), consider $r = o(n)$ and assume $k \geqslant 2$. As $\lambda \to 0$ we may calculate

$$\bar{\eta} = k - 1 + \lambda/k + O(\lambda^2), \quad c = k + \lambda/(k+1) + O(\lambda^2)$$

so that by (16)

$$\mathbf{E}(Y(Y-1)) = c\bar{\eta} = k(k-1) + 2\lambda k/(k+1) + O(\lambda^2)$$

and

$$\mathbf{Var}(Y(Y-1)) = \frac{1}{f_k(\lambda)} \sum_{j \geqslant k} (j(j-1) - k(k-1) + O(\lambda))^2 \frac{\lambda^j}{j!} = \Theta(\lambda).$$

Thus from the definition (2) of η,

$$\frac{\mathbf{Var}\,\eta(\vec{Y})}{(\mathbf{E}\,\eta(\vec{Y}))^2} = \frac{\mathbf{Var}\left(\sum_{j=1}^n Y_j(Y_j-1)\right)}{\left(\mathbf{E}\sum_{j=1}^n Y_j(Y_j-1)\right)^2} = O(\lambda/n).$$

By Chebyshev's inequality and Lemma 1(c), this implies uniformly for n, m, and $\varepsilon > 0$,

$$\mathbf{P}(|\eta(\vec{Y}) - \mathbf{E}(\eta(\vec{Y}))| \geqslant \varepsilon) = O\left(\frac{\lambda}{n\varepsilon^2}\right).$$

Therefore by (22)

$$\begin{aligned}
&\mathbf{E}\left(\exp\left(-\frac{\eta(\vec{Y})}{2} - \frac{\eta^2(\vec{Y})}{4}\right)\middle|\sum_{j=1}^n Y_j = 2m\right) \\
&\quad = \frac{O(\lambda/n\varepsilon^2)}{\mathbf{P}\left(\sum_{j=1}^n Y_j = 2m\right)} + (1+O(\varepsilon))\exp\left(-\frac{\bar{\eta}}{2} - \frac{\bar{\eta}^2}{4}\right) \\
&\quad = (1 + O(\lambda^{1/2}n^{-1/6}))\exp\left(-\frac{\bar{\eta}}{2} - \frac{\bar{\eta}^2}{4}\right),
\end{aligned}$$

upon setting $\varepsilon = \lambda^{1/2}n^{-1/6}$. This gives the form of ξ in (26), recalling $\lambda = O(r/n)$. □

3. Proof of Theorems 2 and 3

We only need to attend to Theorem 3, since Theorem 2 then follows immediately by Theorem 4(a).

Let $a < 1/4$ be fixed. By definition $U_n(\vec{Y})$, defined in (3), is always at most 1, whilst for $\max Y_j \leqslant m^a$, Theorem 1 gives

$$\log U_n = -\frac{\eta(\vec{Y})}{2} - \frac{\eta^2(\vec{Y})}{4} + O(m^{-1+4a}).$$

Since $m = O(n \log n)$, Lemma 1(b) implies $\lambda = O(\log n)$. Thus by (27)

$$\mathbf{P}\left(\max_j Y_j \geqslant m^a\right) \leqslant n\mathbf{P}(Y \geqslant m^a) \leqslant e^{-n^{a'}}$$

for any $a' < a$. So, choosing $a = \varepsilon/4$, the conditional expectation in (13) is

$$O(e^{-n^{a'}}) + (1 + O(n^{-1+\varepsilon}))\mathbf{E}\left(\exp\left(-\frac{\eta(\vec{Y})}{2} - \frac{\eta^2(\vec{Y})}{4}\right)\middle|\sum_{j=1}^n Y_j = 2m\right). \tag{37}$$

Theorem 3 now follows from Theorem 4(b) and (21). □

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