A physicist's view of number partitions

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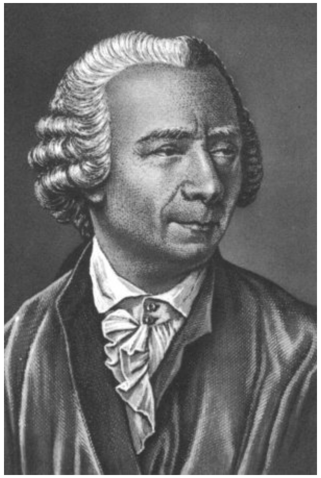
Density of states and Integer Partitions

- Integer Partitions- general.
- Partition function and density of states.
- Bosonic and Fermionic density of states.
- Coloured Partitions!

Integer Partitions- Introduction



At first glance, partitions seem like child's play: For example:

- $$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1+1+1+1$$

  $$p(4) = 5, \quad d(4) = 2$$

- $$5 = 4 + 1 = 3 + 2 = 2 + 2 + 1 = 3 + 1 + 1$$

  $$= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$$

  $$p(5) = 7, \quad d(5) = 3$$

- Soon it gets complicated,

  $$p(200) = 3,972,999,029,388$$

More general partitions

- $$4 = 2^2 = 1^2 + 1^2 + 1^2 + 1^2$$
  $$p^2(4) = 2$$

- $$5 = 2^2 + 1 = 1^2 + 1^2 + 1^2 + 1^2 + 1^2$$
  $$p^2(5) = 2$$

Even more general partitions:
"Bosonic" Partitions

$$p^s(n) \ = \ a_1 n_1^s + a_2 n_2^s + \cdots$$

"Fermionic" Partitions

$$d^s(n) \ = \ m_1^s + m_2^s + \cdots$$

Euler and Ramanujan

Euler's Recurrence Formula:

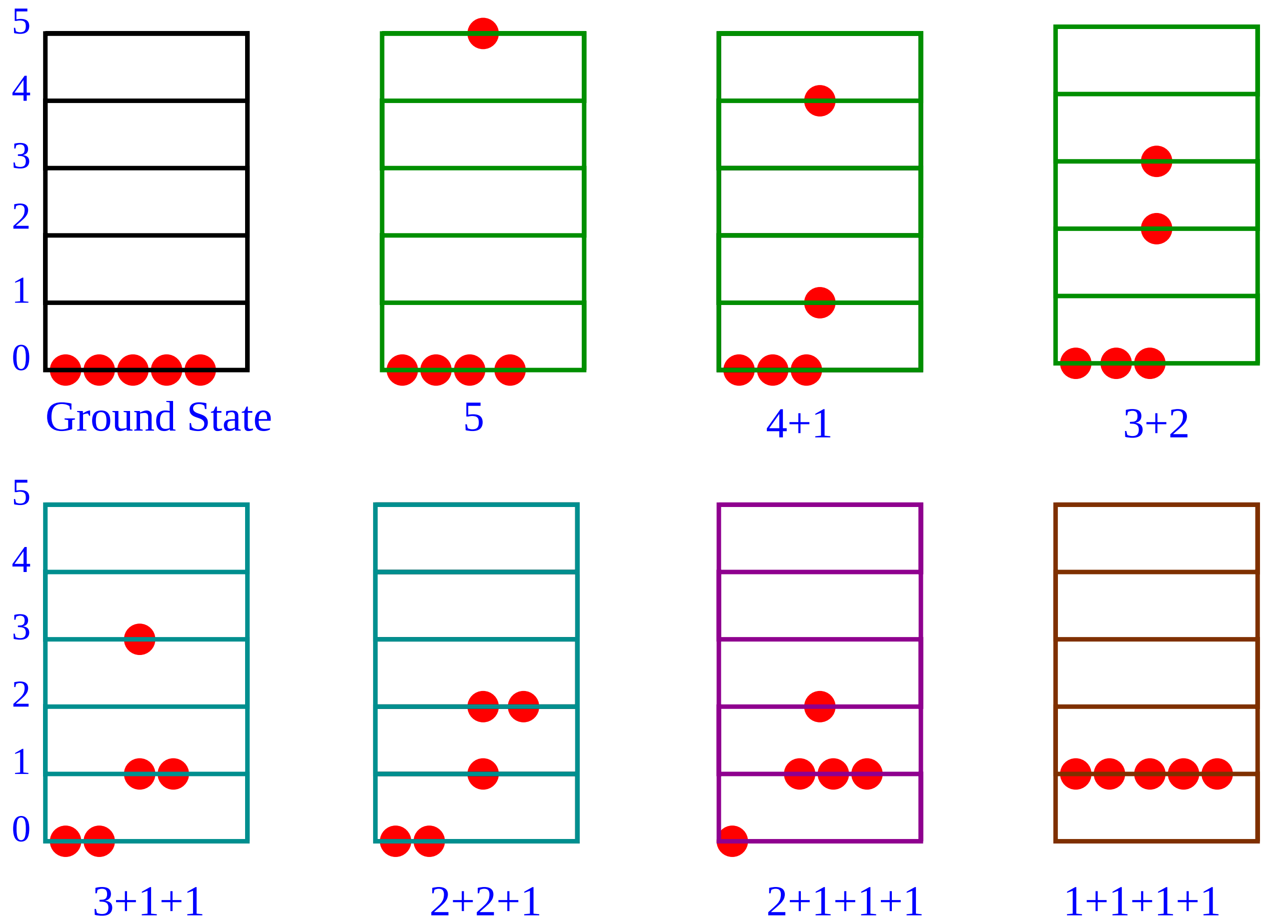
$$
\begin{aligned}
\prod_{n=1}^{\infty} \frac{1}{1-x^n} &= 1 + x + x^2 + x^3 + \cdots \\
&\times \quad 1 + x^2 + x^4 + \cdots \\
&\times \quad 1 + x^3 + x^6 + \cdots \\
&\times \quad \cdots \\
&= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots = \sum_{n=0}^{\infty} p(n) x^n
\end{aligned}
$$

Asymptotic formula for p(n): Hardy and Ramanujan

$$
p(n) \Rightarrow \frac{\exp[\pi\sqrt{\frac{2n}{3}}]}{4\sqrt{3}n}
$$

Quantum Statistics

- Excitations in a systems of Bosons in a Harmonic Trap $p(n)$



Bosonic excitations in Harmonic Oscillator System

- Thus the problem of partitions of an integer into parts is the same as the number of ways of generating the excited states for a given E.

Density of states and $p(n)$

Remarks:

The canonical $N-$particle partition function is given by

$$Z_N(\beta) = \sum_{E_i^{(N)}} D_N \exp(-\beta E_i^{(N)}) = \int_0^\infty \rho_N(E) \exp(-\beta E) dE \ ,$$

Inverse temperature $\beta = 1/T$, N-particle energy–$E_i^{(N)}$

$$\rho_N(E) = \sum_i D_N \delta(E - E_i^{(N)})$$

Number of states available between energy E and E+dE.
That is ρ and $p(n)$ do the same job

$$\rho_N(E) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp(\beta E) Z_N(\beta) d\beta \; = \bar{\rho}_N(E) + \delta\rho(E).$$

In general an oscillating function. Evaluate using the saddle–point method:

$$S(\beta) = \beta E + \log Z_N.$$

Expand the entropy around the stationary point β_0

$$\rho_N(E) = \frac{exp[S(\beta_0)]}{\sqrt{2\pi S''(\beta_0)}} \quad E = -\left(\frac{\partial \ln Z_N}{\partial \beta}\right)_{\beta_0}.$$

This is what we are after! This result is valid for any system, any spectrum and any size.

Bosonic density of states

- The Spectrum

$$\epsilon_m = m^s \quad m \geq 1 \quad s > 0$$

- Canonical partition function as $N \to \infty$

$$Z_\infty(\beta) = \prod_{m=1}^{\infty} \frac{1}{[1 - \exp(-\beta m^s)]}$$

This is the generating function for all excitations (partitions).

- Use saddle-point, Euler-McLaurin expansion, etc: The asymptotic density of states–

$$\rho_\infty^{(s)}(E) = \frac{k}{(2\pi)^{\frac{(s+1)}{2}}} \sqrt{\frac{s}{s+1}} E^{-\frac{3s+1}{2(s+1)}} \exp\left[k(s+1)E^{\frac{1}{1+s}}\right] .$$

This is the celebrated Hardy-Ramanujan formula.

$$C_{(s)} = \Gamma(1 + \frac{1}{s})\zeta(1 + 1/s)$$

$$k_s = \left(\frac{C_{(s)}}{s}\right)^{\frac{s}{1+s}}$$

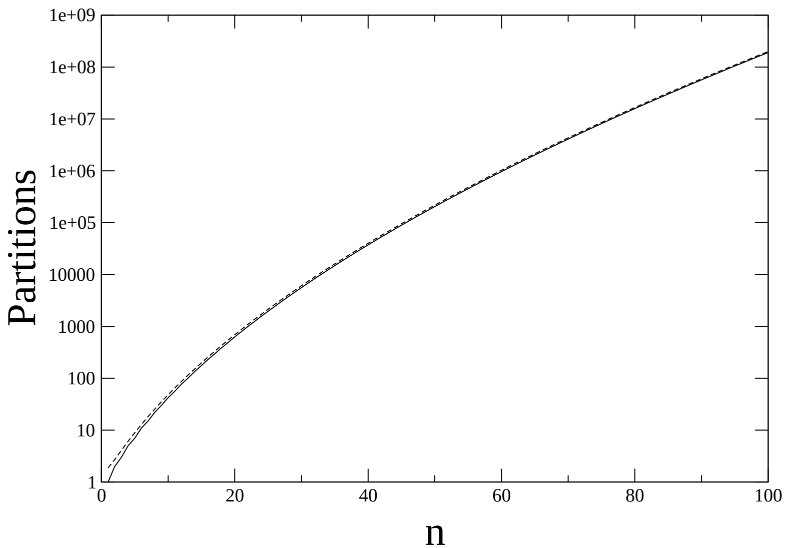
For $s = 1$ we have, for example,

$$\rho_\infty^{(1)}(E) = \frac{\exp[\pi\sqrt{\frac{2E}{3}}]}{4\sqrt{3}E} \Rightarrow p^1(n = E) \quad (1)$$

Same as the density of states of a system of non-interacting fermions in a mean-field–Bethe(1936).

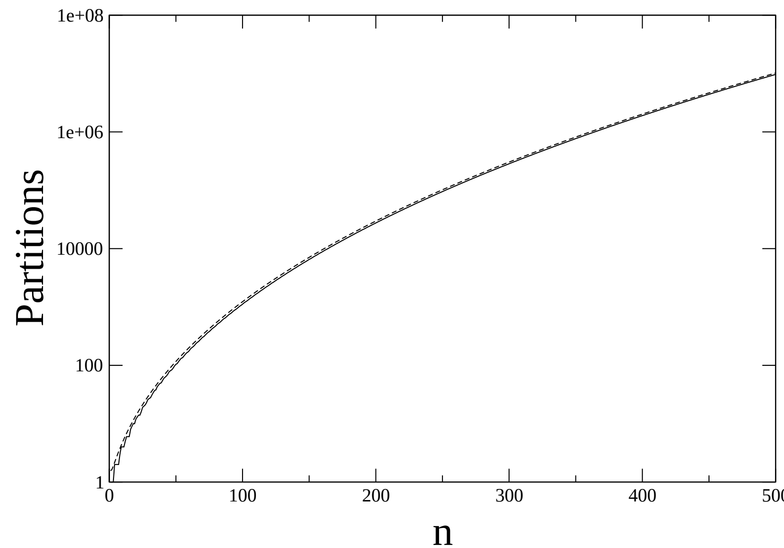
Harmonic Spectrum:

$$s = 1: \ \rho^1(E), \ p^1(n = E)$$



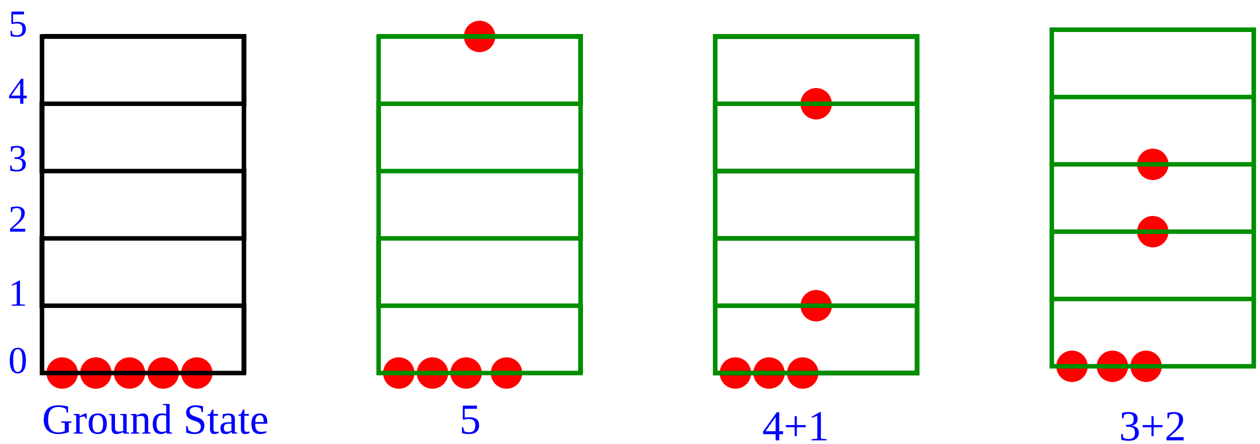
Quadratic Spectrum:

$$s = 2: \ \rho^2(E), \ p^2(n = E)$$



Distinct or Fermionic Partitions

- Collapse of the ground state: Keep the particle distribution imposing Pauli but ignore the hole distribution.



Fermionic excitations in Harmonic Oscillator System

- Partition Function in the limit $N \to \infty$

$$\ln Z_\infty(\beta) = \sum_{m=1}^{\infty} \ln[1 + \exp(-\beta m^s)].$$

GC Partition function but $\mu = 0$ (to collapse the GS).

- Once this change is done, just follow the steps as in the case of bosons:

$$\rho^s_\infty(E) = \sqrt{s\lambda_s}\frac{\exp[(1+s)\lambda_s E^{1/(1+s)}}{2\sqrt{\pi(s+1)E^{(2s+1)/(s+1)}}} \quad \forall \quad s$$

It does not seem to have been derived earlier

$$\lambda_s = \left(\frac{D(s)}{s}\right)^{s/(s+1)} \quad ; \quad D(s) = \Gamma(1+1/s)\eta(1+1/s)$$

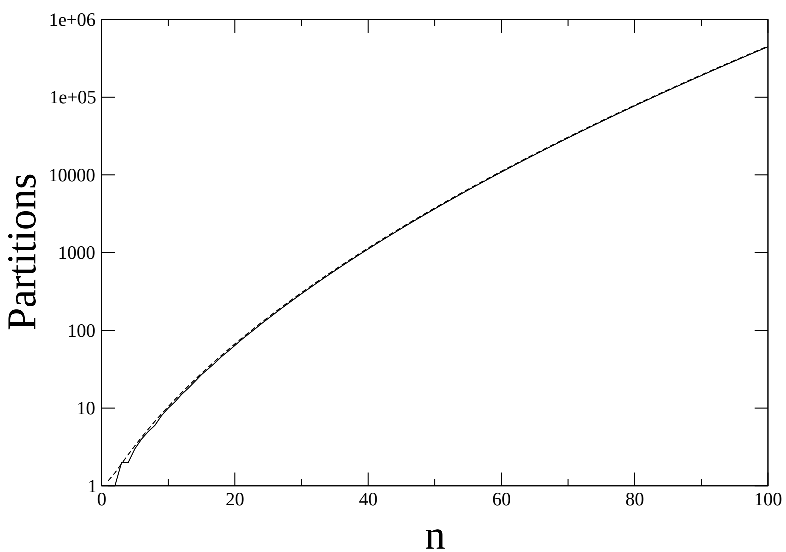
$\eta(s) = \sum_n (-1)^n/n^s$ is the alternating zeta function.

- For $s = 1$ one recovers the famous Ramanujan identity:

$$\rho_\infty(E) = \frac{\exp[\pi\sqrt{\frac{E}{3}}]}{4 \times 3^{1/4}E^{3/4}}$$

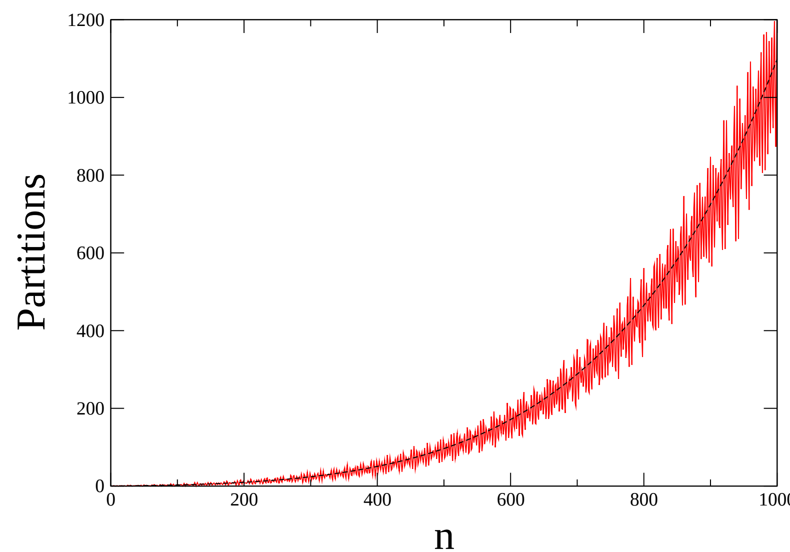
Harmonic Spectrum:

$$s = 1 \; \rho^1(E) \; d^1(n = E)$$



Quadratic Spectrum:

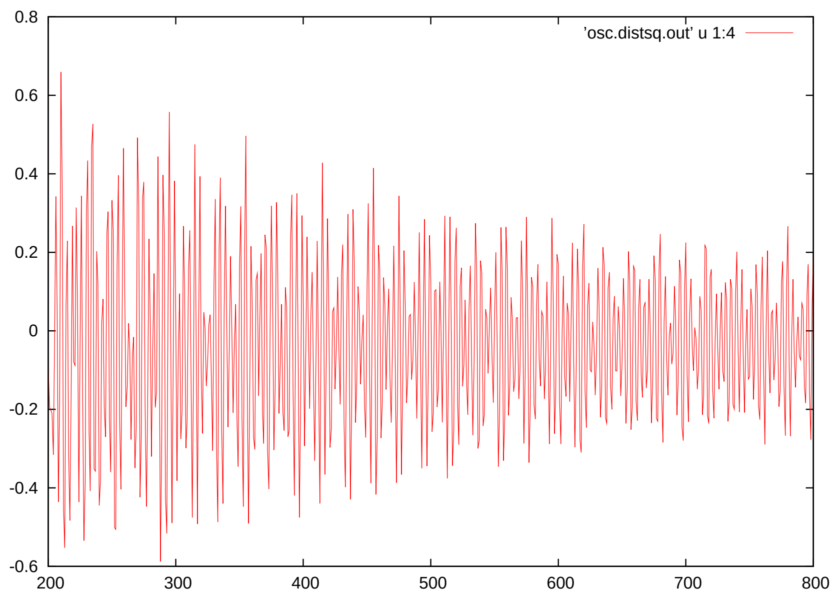
$$s = 2 \; \rho^2(E) \; d^2(n = E)$$

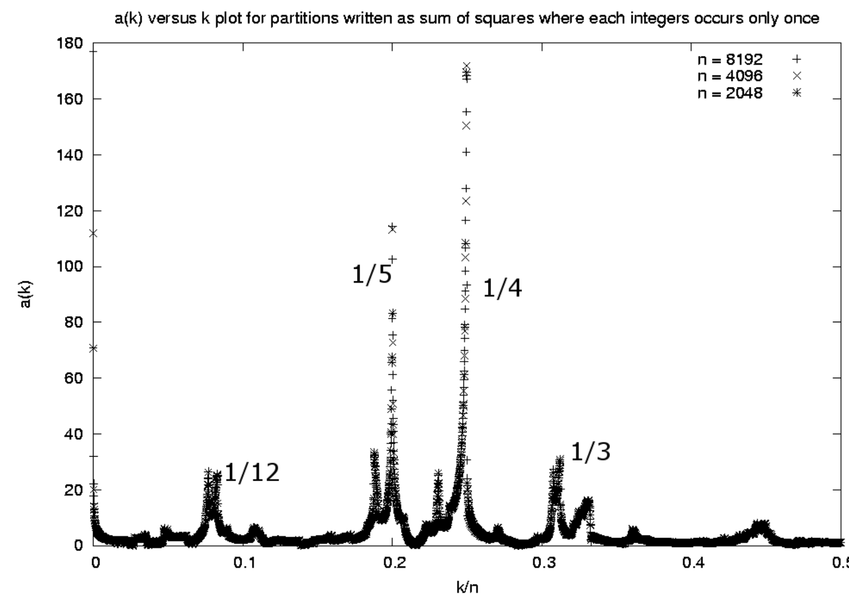


Density oscillations

Quadratic Spectrum: Oscillating part and its Fourier Transform.

$$s = 2 : \ \delta\rho^2(E) = \rho(E) - \rho_\infty(E)$$





Such periods, simple as well as complicated, appear in all partitions with restrictions. (R.Rajesh et al)

Coloured Partitions

We may now push our luck and ask the question "What about all the intermediate partitions"? For example

- Occupancy 1: 5= 5, 4+1, 3+2 (Fermionic)
- Occupancy 2: 5= 5, 4+1, 3+2, 2+2+1, 3+1+1
- Occupancy 3: 5= 5, 4+1, 3+2, 2+2+1, 3+1+1, 1+1+1+2
- Occupancy 5: 5= 5, 4+1, 3+2, 2+2+1, 3+1+1, 1+1+1+2, 1+1+1+1+1 (Bosonic)

Indeed the obvious place to look for are the "Fractional Statistics" systems:

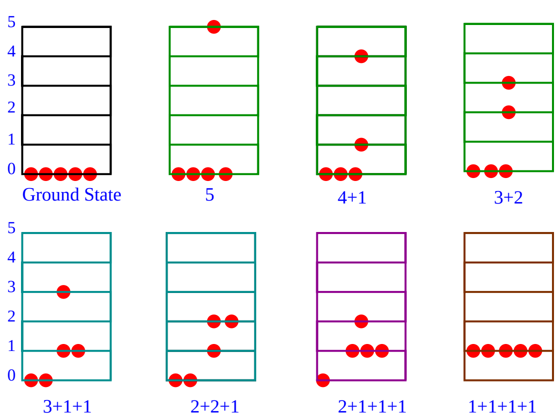
The Institute of Mathematical Sciences

Asymptotic result for coloured partitions

The grand canonical partition function of this system may hence be written as (Gentile, 1939)

$$Z_\infty(\beta) = \prod_{m=1}^{\infty} [\sum_{n=1}^{k} \exp(-n\beta m^s)],$$

where k is the maximum degeneracy.



Bosonic excitations in Harmonic Oscillator System

For $x = \exp(-\beta)$ the partition function may be written as:

$$Z_\infty(x) = \sum_{n=1}^{\infty} p_k^s(n) x^n = \prod_{n=1}^{\infty} \frac{[1 - x^{(k+1)n^s}]}{[1 - x^{n^s}]},$$

Use the Euler-Maclaurin series expansion and saddle point method

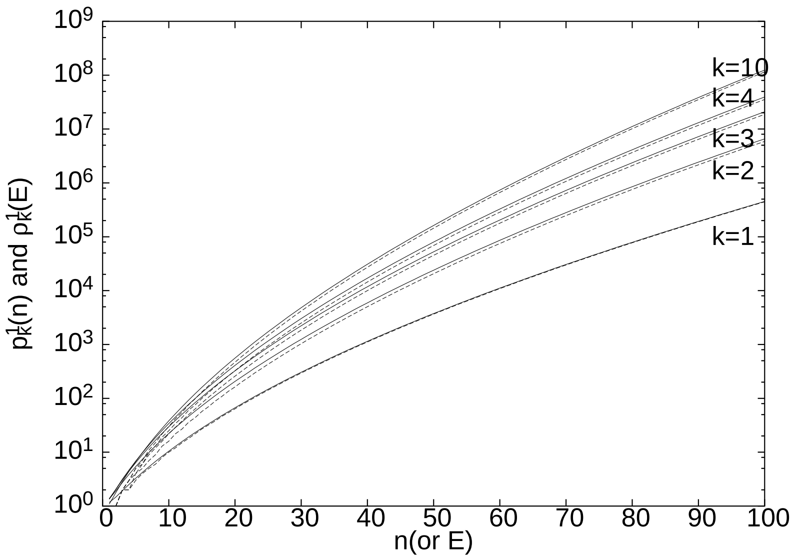
$$S = \beta E + \frac{C(s)}{\beta^{1/s}}[1 - \frac{1}{(k+1)^{\frac{1}{s}}}] - \frac{1}{2}\ln(k+1) + O(\beta)\ ,$$

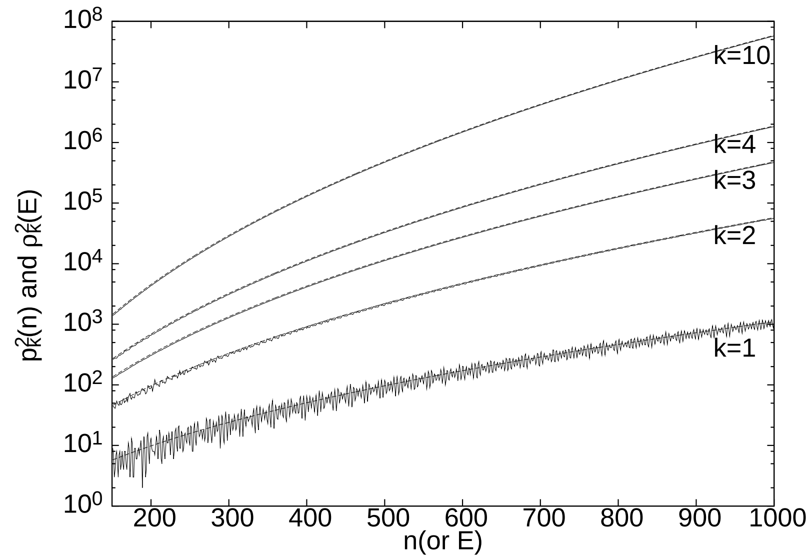
The asymptotic density of states is given by,

$$\rho_{\infty}^{(s)}(E,k)\sqrt{\frac{s\kappa_s}{2\pi(k+1)(s+1)}}E^{-\frac{2s+1}{2(s+1)}}\exp\left[\kappa_s(s+1)E^{\frac{1}{1+s}}\right]\ .$$

$$\kappa_s = \left(\frac{\alpha C(s)}{s}\right)^{\frac{s}{1+s}}, C(s) = \Gamma(1+\frac{1}{s})\zeta(1+1/s),\ \alpha = 1 - \frac{1}{(k+1)^{1/2}}$$

Harmonic Spectrum: Quadratic Spectrum:





Summary

- There is a clear connection between Quantum Statistics and Integer partitions. Many questions remain.
- Understanding the periodicities in some partitions-possible connection to periodic orbit theory?

  On going work with R. Rajesh (IMSc), M.Brack(Regensburg) and R.K.Bhaduri(McMaster).
- Number of odd-partitions of N = Number of distinct partitions (Euler).

  Routes to bosonisation?
- There are many extensions of quantum statistics-Gentile, Haldane, para-bose, para-fermi,....

  What kind of partitions of integers do these systems lead to?