C. R. Physique 13 (2012) 866–877

Contents lists available at SciVerse ScienceDirect

Comptes Rendus Physique

www.sciencedirect.com

Structures and statistics of fluid turbulence / Structures et statistiques de la turbulence des fluides

On the dynamical role of coherent structures in turbulence

Sur le rôle dynamique des structures cohérentes dans la turbulence

Nicholas T. Ouellette

Department of Mechanical Engineering and Materials Science, Yale University, New Haven, CT 06520, USA

A R T I C L E I N F O

Article history:
Available online 3 November 2012

Keywords:
Coherent structures
Turbulence
Fluid mechanics

Mots-clés :
Structures cohérentes
Turbulence
Mécanique des fluides

A B S T R A C T

The notion of coherent structures in fluid mechanics – distinguishable regions of the flow field that share common properties and are correlated in space and time – has played a significant role in characterizing and modeling turbulent flows. They have not yet, however, truly become predictive tools, in part because there is no universally accepted way of extracting coherent structures from the flow field. A wide range of types of structures have been suggested, but there has been comparatively little work done to determine their role in the flow dynamics, and therefore to discern which structures are useful for more than flow visualization. Here, I review several common types of coherent structures, both Eulerian and Lagrangian, with a focus on two-dimensional turbulence far from boundaries. I also discuss a framework for pinpointing the dynamical role of coherent structures based on spatial localization of the spectral properties of the flow. Future work on coherent structures should focus on defining structures that play clear roles in the turbulence dynamics.

R É S U M É

La notion en mécanique des fluides de structures cohérentes, c'est-à-dire des régions particulières de l'écoulement corrélées spatialement et temporellement et qui partagent des propriétés similaires, a joué un rôle significatif dans la caractérisation et la modélisation des écoulements turbulents. Pour autant, ces structures ne sont pas considérées comme des outils prédictifs car, notamment, il n'existe pas de consensus sur la méthode permettant leur extraction de l'écoulement. Il a été proposé une gamme étendue de types de structures, mais peu de travail a été fourni visant à caractériser leur rôle dans la dynamique du fluide, cantonnant donc ces structures à une caractérisation visuelle de l'écoulement. Dans cet article, je me propose de rappeler l'importance de quelques unes de ces structures, aussi bien eulériennes que lagrangiennes, avec un intérêt particulier pour la turbulence bidimensionnelle loin des bords. Je discuterai aussi de l'existence d'un cadre permettant de quantifier leur rôle dynamique en me basant sur la localisation spatiale des propriétés spectrales de l'écoulement. Les études futures des structures cohérentes devront se concentrer à définir les structures qui jouent un rôle clair dans la dynamique de la turbulence.

E-mail address: nicholas.ouellette@yale.edu.

1631-0705/$ – see front matter
http://dx.doi.org/10.1016/j.crhy.2012.09.006

1. Introduction

The "problem of turbulence" is often considered to be one of the grand challenges of classical and statistical physics. But what do we mean by this phrase, and what would a solution to the turbulence problem look like? In one sense, it is already solved: we know the Navier–Stokes equations that govern fluid flow, and in principle they contain all of the turbulence physics. Equations of motion, however, are not particularly useful in applied situations without solutions, and the Navier–Stokes equations cannot be solved for turbulent flows. In addition, since turbulent flows are intrinsically strongly fluctuating, it is natural to describe them statistically; but once we average the equations of motion, they are unclosed [1]. We are therefore forced into modeling, which in turn requires us to deduce the key features of the flow dynamics so that we know what parts of the full Navier–Stokes dynamics we can neglect.

A wide range of approaches to turbulence modeling have been taken, ranging from the very empirical to the very fundamental. But perhaps the most viscerally appealing approach is that based on coherent structures. Here, I use the term "coherent structure" to refer to a connected region of the flow field that displays a macroscopic correlation of some physical property in both time and space. We seek a decomposition of the flow into some kind or kinds of such structures, which will presumably have simpler (or at least slower) dynamics than the whole flow and which we can consider to be the "atoms" of the flow field, and a random background due to intrinsic turbulent fluctuations. The hope is then that a model built only on the measurable properties of these structures can reproduce the essential turbulence statistics. This approach is appealing since many kinds of coherent structures that have been proposed are readily apparent to the naked eye: we are thus attempting to connect what we can *see* to the dynamics of the system. It is also somewhat reminiscent of thermodynamics, where only the self-organized, emergent behavior of the system is necessary to understand it, rather than a detailed characterization of all the degrees of freedom.

As the body of work on coherent structures has grown, so too has the number of types of structures that have been defined. The majority are Eulerian, meaning that they are defined relative to a particular fixed reference frame, and are typically related in some way to local rotation: the prototypical Eulerian coherent structure is a vortex. Although there are various ways to identify vortices in an automated way, most methods have similar aims. As described further below, however, Eulerian definitions of coherent structures are not ideal, and thus researchers have worked to expand the concept of coherent structures in more subtle ways. Recent work has typically focused on Lagrangian information, taken along the paths of fluid elements. Lagrangian structures, as discussed below, are in some sense more coherent than their Eulerian counterparts, since the Lagrangian description implicitly contains information about advection in the flow. Thus, at the cost of less simple visualization, more robust definitions of structures can be formulated.

Simply defining coherent structures, however, is not sufficient. In order to build a model around a particular type of structure, its contribution to the dynamics of the system must be understood. We must thus specify what we mean by "dynamics" for turbulence. One aspect of the turbulence dynamics that we expect to be related to coherent structures is spatial transport and advection. We thus seek a connection between structures and the kinematics of the flow field and particularly of fluid elements. But turbulence does more than redistribute energy and momentum in space in complex ways; due to the nonlinearities in the Navier–Stokes equations, which tend to dominate in developed turbulent flow, energy and momentum are also redistributed between different length and time scales and lead to emergent phenomena such as the Richardson–Kolmogorov energy cascade in three-dimensional turbulence. Ideally, then, we would also like to be able to understand the role played by coherent structures in the scale interactions that are central to our current understanding of turbulence. Few such connections between structure and dynamics, however, have yet been made.

Before continuing, let me mention a few caveats that limit the scope of this discussion. First, I make the assumption that the structures we seek are not examples of pattern formation [2]. Rather, we want non-stationary, fully dynamical structures that will typically only emerge in developed turbulence. Secondly, I will not discuss the wealth of structures that have been documented in turbulent boundary layers and other shear flows [3,4]. Many of these structures, such as the classic hairpin vortices, are intimately connected to the boundary conditions of the flow field. Instead, I will consider the case of homogeneous, isotropic turbulence where the only structures that emerge must be deeply connected to the raw Navier–Stokes dynamics. And finally, much of the discussion below will be focused on two-dimensional turbulence. The detection of coherent structures typically requires knowledge of the full velocity field, with high resolution in both space and time. Such measurements still remain out of reach for high-Reynolds-number experiments in three-dimensional flows. But when the turbulence is confined to a plane, experiments *can* resolve the full velocity field and its gradients in space and time. Additionally, due to the coexistence of a forward enstrophy cascade and an inverse energy cascade, two-dimensional turbulence may display even more self-organization and structure formation than the more usual three-dimensional case [5], making it an ideal test system for studying the dynamical role of coherent structures.

In the remainder of this article, I describe several of the current approaches that have been taken for defining coherent structures. Many of these techniques will be demonstrated with examples taken from laboratory experiments performed in a quasi-two-dimensional flow; thus, I begin in Section 2 by describing the experimental setup. I separate the discussion of coherent structures into those are Eulerian, in Section 3, and those that are Lagrangian, in Section 4. I then address recent approaches to connecting structures with the scale-to-scale dynamics of the turbulence in Section 5, and present a framework for potentially making these links. Finally, I summarize and present some open questions in Section 6.

2. Experimental details

The examples shown in the following all draw on high-resolution data measured in a laboratory quasi-two-dimensional flow. Here, I describe the apparatus and measurement technique used to generate the data.

True two-dimensional flow (that is, flow that is perfectly described by the two-dimensional Navier–Stokes equations) is, of course, impossible to produce in the real world. But we can come close by designing a configuration that has no direct driving of flow along one axis and that will also tend to damp dynamical perturbations in this direction. In practice, these criteria are typically met by making the body of fluid itself highly anisotropic, so that one dimension is very thin compared to the other two; thus, flows must develop primarily in the plane.

Our system consists of a thin layer of fluid, with dimensions of $86 \times 86 \times 0.5$ cm^3; the high aspect ratio of the fluid helps to damp three-dimensional perturbations. But we must take care not to introduce three-dimensional flows in the way we stir the fluid. Since any intrusive stirring protocol, where some kind of device is immersed in the fluid layer, will necessarily drive three-dimensional motion, we instead stir the fluid via a nearly two-dimensional body force; in our case, an electromagnetic Lorentz force. The working fluid in our apparatus is water with dissolved NaCl (16% by mass), and so it conducts electricity. We drive electric currents of up to a few amps laterally through the layer. Beneath the fluid lies an array of permanent magnets, with fields of roughly 3000 gauss at their surfaces. The magnets are laid out in a square grid, with a center-to-center spacing of 2.54 cm. This spacing sets the length scale of the forcing L_f, around which the energy injection is concentrated. For the data shown here, the magnets are arranged in a checkerboard pattern of alternating polarity; other layouts can be used, and produce flows with different dynamical properties [6]. The Lorentz force generated by this arrangement is nearly entirely in the plane, and so stirs the fluid in a two-dimensional way. This type of flow has been used frequently over the past few decades to study mixing and two-dimensional turbulence [7]; our particular apparatus is described in detail elsewhere [8].

The flow field generated by this technique is, of course, not truly two-dimensional. If nothing else, the no-slip condition forces the fluid velocity to vanish on the bottom boundary of the apparatus, even though the fluid above this boundary is moving. Thus, there must be a velocity gradient in the depth direction; for this reason, this flow configuration is often called quasi-two-dimensional. There need not be, however, any significant vertical *flow*. Whenever the in-plane flow field is rotational, there will be some weak but unavoidable Ekman recirculation [9] due to the bottom surface. But although the flow is susceptible to the development of vertical velocities [10], with care the vertical velocities can be negligible [8]. Additionally, we note that given highly resolved velocity fields, any component of the velocity field in the third dimension can be *a posteriori* projected away by a suitable decomposition of the velocity field [8,11]. Nevertheless, the accessible range of Reynolds numbers in this configuration is limited, and truly fully developed two-dimensional turbulence with a very wide range of scales cannot be produced in the laboratory. For the data shown here, the Reynolds number (based on the root-mean-square velocity and the forcing scale L_f) is 185.

To measure the flow, we use particle-tracking velocimetry (PTV). We place fluorescent polystyrene particles with a diameter of 50 μm in the fluid; these particles are small enough that behave as passive flow tracers [12]. Since the particles are lighter than the salt water, they rise to its surface; in this way, all the particles lie at the same depth, so that only one plane is being measured. To avoid any clustering that may occur due to the particles being on a free surface [13], however, we add a layer of pure water (also about 5 mm deep) on top of the salty layer. The miscibility of the salt water and pure water means that there is no bulk surface tension between them, so that the particles are not attracted to one another.

The particles absorb blue light and fluoresce in the green. We use inexpensive but bright blue LEDs for illumination. The fluorescence signal is recorded at a rate of 60 frames per second, a sufficient speed, given that typical root-mean-square velocities in this system are of order centimeters per second. We define the eddy turnover time T_L, which is approximately the time scale associated with the forcing, as $L_f/\langle u^2\rangle^{1/2}$; it is typically on the order of a few seconds. We record images for hundreds of T_L using a digital camera with a resolution of 2320×1728 pixels. We image only the central 32×24 cm^2 (roughly $12L_f \times 9L_f$) of the apparatus, so that any effects of the lateral walls on the flow are not present in our field of view. We locate the particles in each image using one-dimensional Gaussian fits, and match their positions together in time to create particle trajectories using a three-frame predictive algorithm [14]. Once trajectories have been constructed, information from multiple time steps can be used to compute accurate velocities; we measure the velocities by convolving the trajectories with a Gaussian smoothing and differentiating kernel [15]. Finally, we generate velocity fields by projecting the measured data on a basis of numerically computed streamfunction eigenmodes [8].

3. Eulerian structures

The traditional definitions of coherent structures are Eulerian: we seek objects that are defined relative to some fixed (laboratory) coordinate system and that typically only involve instantaneous information about the flow field. This approach is not surprising. Eulerian analysis has been the standard tool of fluid mechanics for decades, as Lagrangian experiments and simulations have only become possible comparatively recently. And, moreover, since much of the appeal of a coherent-structure analysis comes from the desire to connect what we see with the statistics we measure, the close relationship between Eulerian structures and flow visualization [4] makes the approach natural.

As the most visually striking flow patterns tend to be regions of strong rotation, most techniques for determining Eulerian coherent structures seek to find vortices. We discuss methods for locating vortical regions below. Subsequently, we discuss

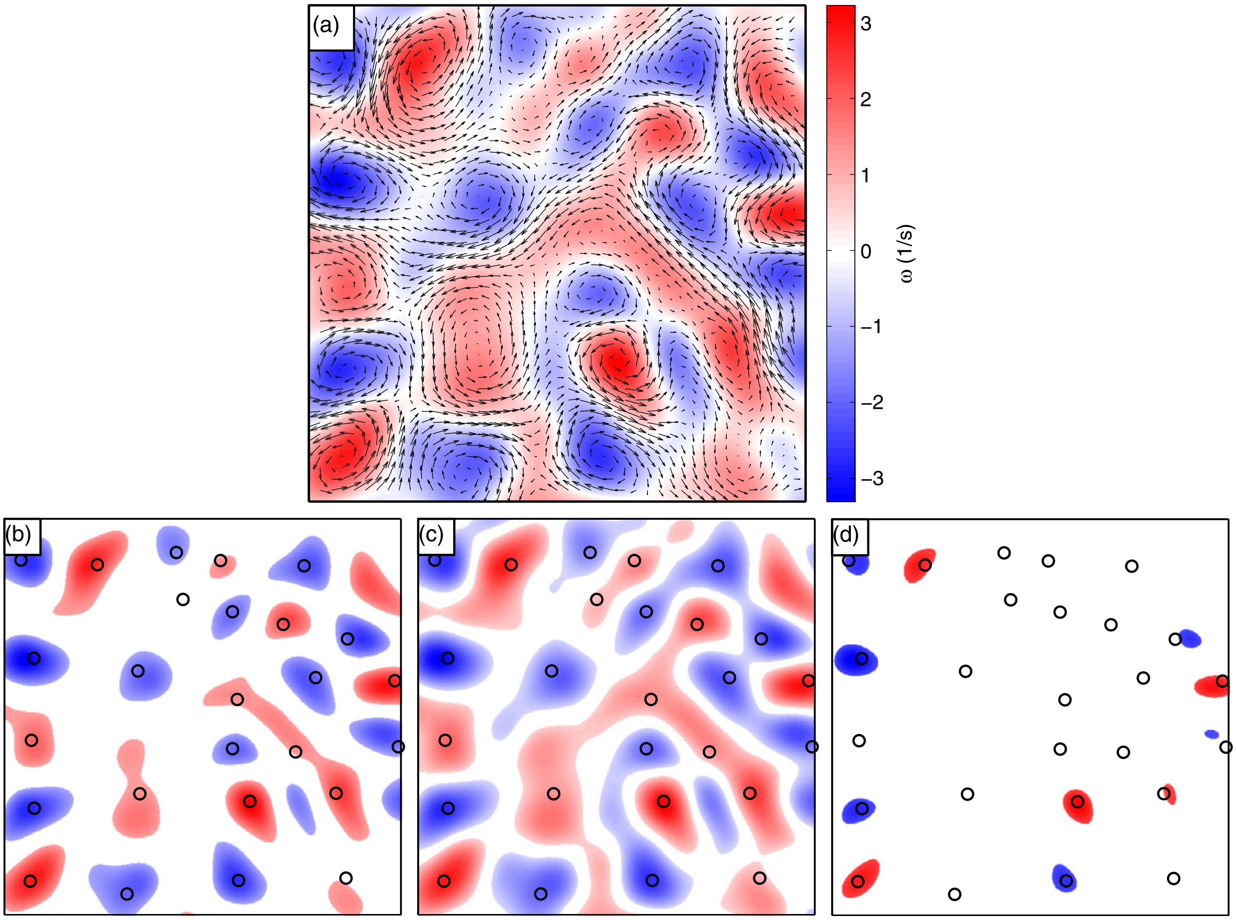

Fig. 1. (a) Full velocity (arrows) and vorticity (shading) field for a subregion of our flow field measuring roughly $6L_f$ in each direction. (b–d) Thresholded "large" vorticity regions for the same velocity field. The thresholds are chosen as (b) $\langle\omega^2\rangle^{1/2}$, (c) $\langle\omega^2\rangle^{1/2}/2$, and (d) $2\langle\omega^2\rangle^{1/2}$. Vortex cores are shown as black circles.

tools that have been developed to identify the vortex cores themselves, as well as other point-like Eulerian structures. Finally, we discuss some fundamental concerns with all Eulerian definitions of coherent structures, which will lead us to consider Lagrangian structures instead.

3.1. Extended rotational regions

If we are going to seek vortices, we must first define what we mean by a vortex. The simplest choice is to look for regions of "high" vorticity [4,16,17]. Unfortunately, there is no way to tell *a priori* how large vorticity must be in order to separate coherent vortices from the background flow field [18]. As an example, Fig. 1 shows the full vorticity field as measured in our experiment as well as regions chosen by selecting only vorticity larger than some threshold – in this case, the root-mean-square (rms) vorticity, half the rms, and twice the rms.

For each choice of threshold, the resulting pattern is different. As one would expect, larger thresholds lead to smaller, more disconnected regions, while smaller thresholds lead to large "blobs" of vorticity. In each image, we have also included the independently identified vortex cores, defined to be the instantaneous elliptic critical points in the flow field (see below for more details). For no threshold do we find a one-to-one correspondence between single vortex cores and extended large-vorticity regions. An extended vortex may contain more than one core or none at all, or a vortex core can lie outside a large-vorticity region. Such cases are likely to occur when vortices are born or die and the magnitude of their rotation changes significantly – but these are situations that one would certainly like to capture with a coherent-structure analysis, since the creation or disappearance of structures is likely to be crucial to a complete analysis! In addition, the boundaries of vortices identified by a thresholding technique have a vanishingly small likelihood of being material surfaces. Thus, from the point of view of a fluid element or, more generally, from the perspective of transport and advection, it is difficult to ascertain the effect of one of these vortices.

Problems such as those we have illustrated here plague all Eulerian definitions of extended vortices, whether they are defined via simple thresholding as we have done here or by other more complex criteria [18–20]. And yet there does seem

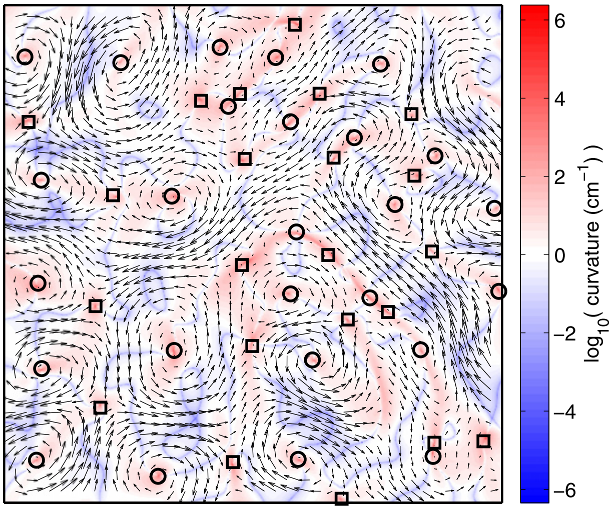

Fig. 2. Curvature field (shading), velocity field (arrows), and critical points (symbols) for the same data as in Fig. 1. The shading shows the logarithm of the curvature, as it has a tremendously large dynamic range [28,29,31]. Hyperbolic points (in straining regions) are shown with squares, while elliptic points (in rotational regions) are shown with circles.

to be some statistical signature of “vortices”, or at least of strong rotation, that can be identified for turbulent flows. For example, Biferale et al. [21] showed that filtering out rapidly rotating vortices (which they identified via strong acceleration events, a measure that may also exclude other types of flow features) modified the flow intermittency. Their results were echoed by a subsequent study that used particles with finite inertia, which are expected to be expelled from vortical structures, to achieve the filtering [22]. The essential contributions of intense vortical regions to turbulence and intermittency, however, remain somewhat unclear [23]; they may be more an effect than a cause.

3.2. Critical points

Although defining a spatially extended vortex is somewhat arbitrary, one *can* exactly define its core. Suppose one drew an imaginary line that passed perpendicularly through a vortex core. On one side of the core, the velocity field along this line would point in one direction, while on the other side, the velocity must point in the opposing direction, so that there is a net circulation around the vortex. Assuming that the velocity field is smooth, the intermediate value theorem thus guarantees that there must be a point along the line where the magnitude of the field vanishes – and we can define this point as lying on the vortex core. Note that these cores do not necessarily lie at the local maxima of the vorticity field.

Stagnation points such as vortex cores have special topological significance. They can act as starting and ending points for streamlines in the bulk of the flow field, and their locations and the local flows around them act as a skeleton that determines much of the overall flow structure [24]. In dynamical systems theory, where they are often known as critical points, these stagnation points can be related to chaotic mixing [25]. In some ways, they are reminiscent of the defects that are often studied in pattern-forming systems [26]; here, however, they are integral components of the velocity field rather than defects per se. In 3D incompressible flow, five types of critical points are possible [27]. In 2D incompressible flow, however, only two kinds are allowed: elliptic points (located at vortex cores) and hyperbolic points (located in purely straining regions).

Recently, Ouellette and Gollub [28,29] developed a robust method for locating critical points in experimental data sets. The simplest way to locate critical points is to look for zeros of the Eulerian velocity field; and indeed, if only single-time information is known, this is the only method that can be applied. But since modern particle-tracking techniques can provide the entire Lagrangian history of each particle, this additional information can be used in data analysis schemes. In the case of critical points, we rely on the observation that at a stagnation point, either elliptic or hyperbolic, the trajectory of a fluid element will tend to change direction significantly over a short distance in space – and thus, critical points can be identified as points in the flow that locally maximize the instantaneous curvature of a fluid-element trajectory. The curvature of nearby trajectories must be similar as long as the velocity field is smooth, and thus we can construct an Eulerian field of this Lagrangian quantity. Empirically, the maxima of this field are both very sharp and very large compared to the background values, and so can be located with more precision than zeros of the velocity field [28,29]. Thus, even though the statistics of the curvature itself do not contain a tremendous amount of information about the flow [30,31], the spatial structure of the curvature does. In order to verify that the resulting points are indeed critical points, and to classify them as hyperbolic or elliptic, we measure the topological charge [32] of each candidate point.

In Fig. 2, we show the hyperbolic and elliptic points located by this technique in the same velocity field shown in Fig. 1. At a single instant of time, the locations of these points do not give us much information about the kinematics of the flow field. But as the velocity field changes in time, so do the locations of the critical points – and following their motion can give us a low-dimensional way to characterize changes in the field. Ouellette and Gollub, for example, found that interactions between hyperbolic and elliptic points, in the form of pair creations and annihilations that conserve global topological charge, occur only above a critical Reynolds number, which they identified as the onset of spatiotemporal chaos [28]. They found that this critical Reynolds number was tied to the spatial structure of the flow forcing, dropping as the flow forcing became more disordered. The dynamics of critical points at higher Reynolds numbers, when the flow is strongly turbulent, have not been studied.

Finally, we note that work has also been done on the critical points of the acceleration field, also known as ZAPs (for zero-acceleration points) [33]. Like the hyperbolic and elliptic points described above, ZAPs are topological objects, and move differently from fluid elements. Because the acceleration field is not constrained to be divergence-free, there are three types of ZAPs even in two-dimensional flow, and their interactions are somewhat more complex than those of velocity critical points. ZAPs in rotating and straining regions of the flow field cannot annihilate directly, and thus an imbalance of the two types can be created as the flow evolves; ZAPs in rotating regions tend to live longer [33], which may partially explain why it is so easy to observe vortical structures. Although more work remains to be done to appreciate the importance of ZAPs fully, they may have potential use in segmenting the flow field into different types of regions [33] or in characterizing the transport of inertial impurities in the flow [34].

3.3. Frame dependence

All Eulerian coherent structures share some common shortcomings that make them less than ideal for our goal of defining a structure with a clear connection to the flow dynamics. This is not to say that they are not useful for gaining a qualitative understanding of the flow; indeed, flow visualization can be very valuable. But it is unlikely that we will ever be able to develop a faithful and predictive model of the flow based solely on Eulerian structures.

The most serious shortcoming of Eulerian structures is that they are necessarily tied to a given reference frame. Some types of Eulerian structures are more tied to their frame than others. Critical points, for example, are not invariant under Galilean boosts. Criteria for determining vortices can be designed to be Galilean invariant [19], but are typically not invariant under non-inertial reference frame shifts [18] – an important consideration in rotational flows, such as those with strong vorticity. In some cases, there is certainly a preferred reference frame, such as, for example, the rest frame of the wall in a boundary-layer flow. But in homogeneous, isotropic turbulence, there is no such preferred frame, and so we would like to define coherent structures that are frame-independent. In order to make progress, then, we must define our structures with respect to frame-independent quantities. Luckily, fluid flows provide us with a natural choice: the trajectories of fluid elements. In the following section, we describe methods for defining coherent structures based on this Lagrangian information.

4. Lagrangian structures

Many of the fundamental problems with Eulerian coherent structures can be solved by positing new types of coherent structures defined in the Lagrangian framework. Since these structures must involve the trajectories of fluid elements in some way, they are by definition independent of reference frame: regardless of our vantage point, the fluid elements all move along trajectories that are well defined curves in space. Even though the representation of these curves changes in different reference frames, the trajectories themselves are invariant. Additionally, the contribution of Lagrangian structures to transport is typically easier to understand, since advection can be much more naturally captured in a Lagrangian sense. Since Lagrangian structures inherently involve spatiotemporal information, they are in some ways more coherent than Eulerian structures, where the temporal behavior is not as deeply linked.

Shifting to Lagrangian definitions of coherent structures is not a panacea, however. One issue is that they require more data to compute, since they must involve the time evolution of the flow. This requirement makes them nearly impossible to locate in many situations. Additionally, since they tend to be more abstract than, say, simple connected regions of large rotation, Lagrangian structures lack the visceral appeal of Eulerian structures, and are less clearly linked to what we see when we observe the flow visually. Nevertheless, Lagrangian structures may hold more promise for actually linking back to the flow dynamics.

4.1. Multipoint Lagrangian clusters

The fundamental quantity of interest in the Lagrangian description of fluid mechanics is the trajectory of a fluid element. But a single trajectory hardly fits our conception of what a “structure” should be. Thus, we are led to consider not single Lagrangian points but rather *clusters* of points. In particular, it is reasonable to define a minimal Lagrangian structure as a cluster of the minimum number of fluid elements needed to span the space – four in three-dimensional space, or three in two-dimensional space. We can then think of these clusters as simple, tractable approximations of material volumes or areas. This approach is the basis of the well known “tetrad” stochastic model for 3D turbulence [35].

In homogeneous turbulence, the absolute location of these clusters is statistically irrelevant. Instead, it is the relative position of the particles that make up the cluster that is important; in other words, the *shape* of the cluster is the key quantity [36]. The distortion of an ensemble of initially isotropic clusters to some other distribution of shapes gives clues towards the dynamics of asymmetric processes in turbulence (such as energy dissipation), and may be related to intermittency [37] and the dynamics of the energy cascade [38]. The stationary distributions, as well as the scale-dependent rate at which clusters approach these distributions, have been studied in fully developed, 3D turbulence both experimentally [39] and numerically [40], with similar results. Recent work on Lagrangian clusters in 3D has explored the dynamics of angular momentum in the flow [41].

Study of Lagrangian clusters in 2D turbulence has been somewhat more detailed, since it is easier to track several points over the long times required to watch their evolution in 2D. Just as in 3D flow, clusters of any initial shape distribution will evolve towards a universal, stationary shape distribution. But an additional feature of the dynamics arises in 2D that has not been observed in 3D studies: as the ensemble evolves towards its final shape distribution, it first "overshoots" the final distribution and displays a transient distribution of highly distorted shapes [42]. Although this phenomenon has been interpreted as a signature of turbulent cascades [43,44], we also observed it at lower Reynolds numbers where there is no developed turbulence [42,45]. Intriguingly, the overshoots are only present in actual fluid flows; simulations of randomly diffusing particles do not show them, even though the final shape distributions are identical [45]. Thus, it appears that the path taken by the ensemble through the phase space of possible shapes encodes information about the nature of the flow field. How this information can be extracted and parsed, however, remains an open question.

As we have described here, it is primarily the statistics of Lagrangian clusters that have been studied – that is, the properties of an ensemble of clusters. In that sense, they are not really coherent structures as we defined them above, because no direct connection is made to the spatial structure of the flow field. Although clusters certainly span some connected region in space, that region has no special properties. Instead, the selection of the initial positions of the points in the cluster is typically taken to be random and arbitrary. Some work has been done to attempt to connect the shape evolution of clusters to the instantaneous stagnation-point structure of the flow [45] (with similar work having been done for the lower dimensional two-particle case [46]), but clusters are still treated primarily statistically.

4.2. *Lagrangian coherent structures*

Perhaps the leading current candidates in our quest for a structure that is both well defined and dynamically relevant are known as Lagrangian Coherent Structures (LCS). Unlike many of the structures described above, LCS are not at all ad hoc. Rather, they are a clear extension of established methods from dynamical systems theory – and Lagrangian fluid mechanics can of course be considered to be a dynamical system, albeit one that is extremely complicated, with infinite dimensionality and no time periodicity. To understand the behavior of a low-dimensional dynamical system, one studies the structure of the stable and unstable manifolds of hyperbolic fixed points. As first introduced by Haller [47–49], LCS are essentially a generalization of this concept to the high-dimensional, aperiodic dynamics that characterize complex fluid flow.

In the context of fluid mechanics, LCS are special material lines in the flow. Since they are the finite-time analogs of stable and unstable manifolds, we seek material lines that have similar properties; namely, we seek the material lines that attract or repel fluid elements more strongly than do their local neighbors. These material lines, which are the LCS, give us the "skeleton" of advection in the flow [50]; as the most important organizing elements of the flow, they can give a qualitative picture of where fluid elements will go over some interval of time in the future. In addition, since they are material lines, they are transport barriers for fluid elements or passive scalars [50,51]. Thus, unlike the coherent structures discussed above, LCS have a clear and understood connection to (at least) the flow kinematics. They have been shown experimentally to determine the mixing structure of passive scalars [52], and have begun to find application in analyzing large-scale geophysical flows [53–56].

Locating LCS is somewhat nontrivial, and requires a significant amount of information about the flow field. In particular, in principle they require knowledge of the *future* evolution of the flow field: in order to determine which material lines have the necessary properties, we need to know how they interact with fluid elements. Despite this strong requirement, however, LCS have still proved to be a useful *a posteriori* way to analyze complex fluid motion; and indeed, recent work has begun to relax the amount of future information needed [57].

Many methods have been proposed for locating LCS. But typically, LCS are determined by studying the field of finite-time Lyapunov exponents (FTLEs), although more recent work has suggested refinements to this method that avoid some possible false positives [58,59]. To make this discussion more concrete, several FTLE fields from our 2D flow are shown in Fig. 3 for the same instant of time shown in Fig. 1. To calculate these FTLE fields, we follow the method outlined by Voth et al. [52]. We first define a grid of "virtual" tracer particles, which we then advect through our measured velocity fields by solving their equations of motion numerically for a time T [12]. Using the measured trajectories of real tracers is also possible, but leads to noisier and less well resolved results, since the tracers are located randomly in the flow field and their trajectories are all of different lengths. We then determine the displacement of each virtual particle over T; these displacements allow us to compute the flow map $\boldsymbol{\Phi}(\mathbf{x}, t, T)$, a vector field that maps the position $\mathbf{x}$ of a fluid element at time t to its resulting position at time $t + T$. Taking the inner product of the gradient of the flow map with itself gives the Cauchy–Green strain tensor

$$C_{ij} = \frac{\partial \Phi_k}{\partial x_i}\frac{\partial \Phi_k}{\partial x_j} \tag{1}$$

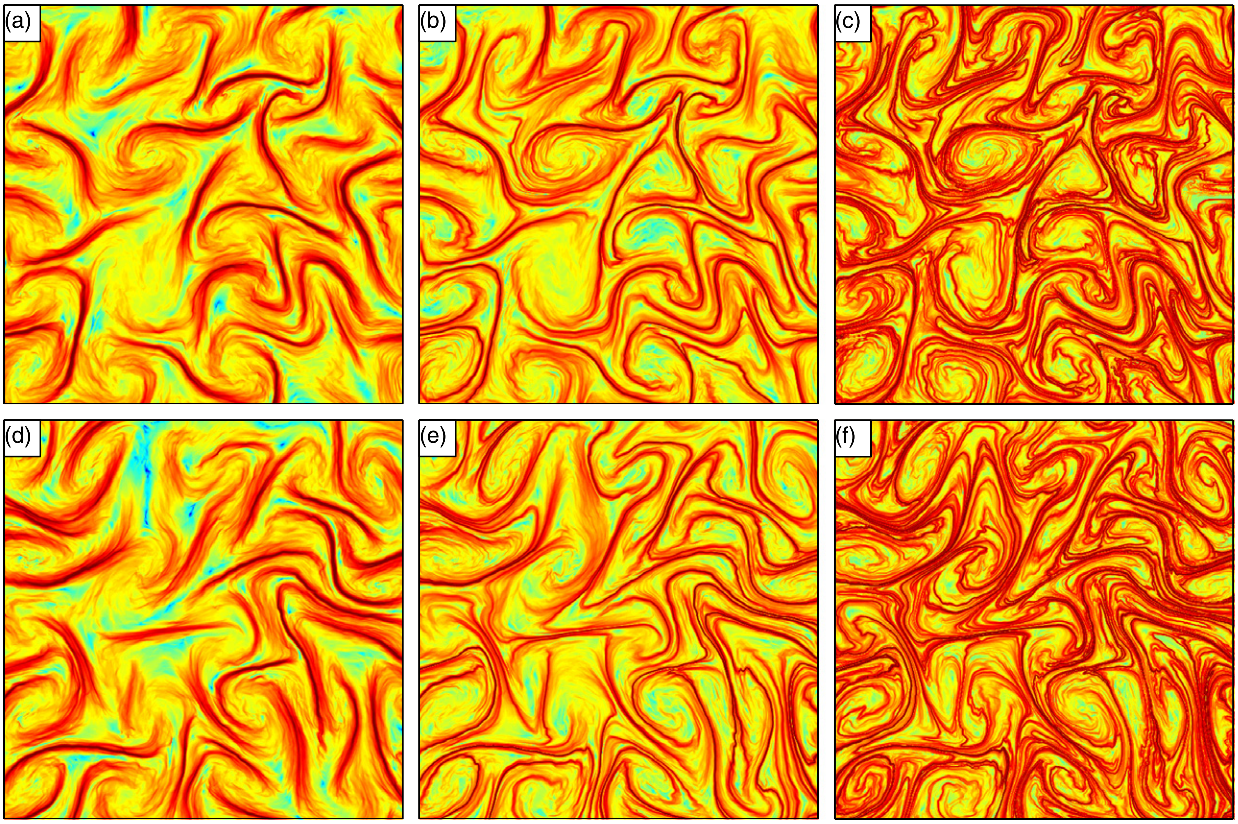

Fig. 3. Finite-time Lyapunov exponent (FTLE) fields for the same data as in Fig. 1. The top row (a–c) shows forward-time FTLEs; the bottom row (d–f) shows backward-time FTLEs. From left to right, the time lag T over which the FTLEs are computed increases, from $T = T_L$ (a, d) to $T = 2T_L$ (b, e) to $T = 4T_L$ (c, f). The ridges of these FTLE fields are Lagrangian coherent structures.

where summation is implied over repeated indices. Note that the object defined here is the right Cauchy–Green tensor; if we had contracted the other set of indices, we would have defined the left Cauchy–Green tensor. The two have the same eigenvalues, though the eigenvectors are different [60]. The FTLE $\sigma(\mathbf{x}, T)$ can then be obtained as

$$\sigma(\mathbf{x}, T) = \log\left(\sqrt{\lambda_{\max}(\mathbf{x})}\right)/T \tag{2}$$

where $\lambda_{\max}$ is the largest eigenvalue of C_{ij}. The quantity $\sqrt{\lambda_{\max}}$ itself is also used to study fluid flow [52], and gives the stretching of the fluid element initially at $\mathbf{x}$ over the time T. As shown in Fig. 3, the FTLE field sharpens and gains structure as T increases, though the gross shape does not change. Also, the time T need not be positive; we can also consider the *backward* FTLEs, extracted in the same fashion but for negative T. LCS computed in negative time are repelling structures, while those computed in positive time are attracting [52].

For both the forward and backward cases, common features are clear in Fig. 3. The fields are composed of areas where the FTLEs are small (and therefore fluid elements stretch only a small amount) separated by line-like regions where the FTLEs are large. These ridges of the FTLE field are taken to be the LCS, and separate regions of the flow that are kinematically independent, in that over the lifetime of the LCS they cannot exchange energy and momentum via advection. These regions separated by LCS can thus themselves potentially be considered to be coherent structures, since they are bounded from each other by impenetrable barriers. And, appealingly, both the LCS and the regions in between are defined in a Lagrangian way, so that they are independent of reference frame and do not suffer from some of the problems that plague Eulerian structures.

The technique described here only strictly applies to fluid elements in a smooth velocity field, since the flow map and Cauchy–Green tensor are essentially a linearization of the flow dynamics. Thus, when applied to turbulence, it should be used to characterize only the smallest scales of the flow, where the deformation of a material volume is small and therefore approximately linear, and may not fully describe the inertial-range dynamics. But the basic underlying idea of looking at the (Lagrangian) deformation of a material volume can be generalized to macroscopic volumes that deform in complex, nonlinear, non-affine ways [61]. Suppose we parameterize a material volume by its centroid $\mathbf{x}^{(0)}$ and a number of points on its boundary $\mathbf{x}^{(n)}$, much as we did above when we discussed Lagrangian clusters. We define $\mathbf{d}^{(n)} \equiv \mathbf{x}^{(n)} - \mathbf{x}^{(0)}$ as the distance from a boundary point to the centroid. As the material volume changes its shape, these distances all change. But if the deformation is simple and affine, all the distances change in a coherent way that can represented by the action of a linear matrix operator. Deviations from this behavior must be caused by non-affine, nonlinear effects. We can extract the net non-affine deformation by performing a least-squares fit of the best affine operator that describes the deformation and looking at the residual. That is, we compute

$$D^2(\mathbf{x}, t, T) = \min_{\alpha_{ij}} \left[\frac{1}{d_0^2 N} \sum_{n=1}^{N} \left(d_i^{(n)}(t+T) - (\delta_{ij} + \alpha_{ij}) d_j^{(n)}(t) \right)^2 \right] \tag{3}$$

where d_0 is the initial average magnitude of the distances from the centroid of the volume to its boundary, N is the number of points we are using to parameterize the volume, δ_{ij} is the identity tensor, and α_{ij} is the best-fit affine deformation operator. This technique is similar to that used to detect shear transformation zones in glassy systems [62]. D^2, the residual of the fit, gives us a measure of the non-affine deformation. As shown by Kelley and Ouellette [61], D^2 becomes appreciable when the material volume is macroscopic and T is not infinitesimal. It has a spatial structure similar to the FTLE field, but the magnitudes can be quite different; an explicit comparison of D^2 and the FTLE field is shown in Ref. [61]. This tool may be more appropriate for searching for Lagrangian structures in the inertial range.

LCS and their extensions are thus excellent candidates for structures that we may be able to link back to their role in the dynamics of the flow field. But despite their promise, LCS have been studied very little in 3D turbulent flows. This may be in part due to the very high computational cost associated with determining the FTLE field via the method given above [63]. Additionally, the full 3D, time-resolved velocity field is required but is typically not yet possible to obtain in experiments. Further advances, both computational and experimental, are likely to make the determination of LCS possible in 3D turbulence, and may lead to new discoveries.

5. Scale-to-scale dynamics

The generation of and coupling between disparate length and time scales due to the strong nonlinearities in high-Reynolds-number flow are central to our understanding of turbulence. Indeed, a practical working definition of "turbulence" might be a flow that has an energy cascade and shows an energy spectrum that scales like $k^{-5/3}$ (where k is a wavenumber) for some appreciable range. But quantities like the energy spectrum (or indeed the cascade itself) that are well defined only in Fourier space are difficult if not impossible to relate to the kinds of coherent structures we have been discussing. And yet historically researchers have always described the activity at different scales in language that is suggestive of structure: Richardson, for example, wrote of "whirls" in his classic poem [64], and the usual language of "eddies" supposes that there is some spatial coherence to the partitioning of energy among length scales. It is desirable, then, to link any coherent structures we define in turbulent flow to their role in the energy cascade – essentially, to specify how they are linked to the classical picture of an eddy.

Making these links is not trivial, since the energy cascade is defined in Fourier space while coherent structures are defined in real space. We thus need a tool that lies somewhere in between: we seek a way to localize the interaction between scales in real space. Although we can never accomplish this goal perfectly (due to unavoidable uncertainty relations), we can make significant progress by applying filtering techniques [65]. The idea is somewhat similar to the approach of Large Eddy Simulation (LES); unlike LES, however, we will not need to model any unresolved scales, but rather will make direct measurements. The tools we describe here [66,67], sometimes called "filter-space techniques" (FSTs), have begun to see significant use in recent years, and have been particularly useful for studying the direct enstrophy and inverse energy cascades in 2D turbulence [68–71]. Here, we introduce them as a framework for investigating the relationship between coherent structures, which are by their nature spatially localized, and the scale-to-scale dynamics of the turbulence.

The idea of an FST is simple, if somewhat counterintuitive: we seek to understand more about the flow by intentionally throwing away information. To do this, we will remove the high-wavenumber components of the velocity field by convolving it with a function that acts as a low-pass filter in Fourier space. Let us define this filtered field as

$$u_i^{(r)}(\mathbf{x}) \equiv \int G^{(r)}\left(\mathbf{x} - \mathbf{x}'\right) u_i\left(\mathbf{x}'\right) \mathrm{d}\mathbf{x}' \tag{4}$$

where G is the filter kernel. Many kernels have been used in the literature [1]; Gaussian kernels are typically used in this context for simplicity and since they are well behaved in both real and Fourier space [68]. The superscript (r) in our notation signifies a filtered quantity with all length scales smaller than r removed; and note that, since we in principle begin with the fully resolved field, we can take r to have any value we wish. Example filtered velocity fields (using a Gaussian filter kernel) are shown in Fig. 4.

In order to write the equation of motion for the filtered velocity, we apply the filtering operation to the Navier–Stokes equations. When we do so, we find that filtering the nonlinear term leads to an extra contribution to the equations that looks like a stress – much as the Reynolds stress emerges after applying a Reynolds decomposition to the velocity field [1]. This new term, known as the sub-grid stress in LES, has the form

$$\tau_{ij}^{(r)} = (u_i u_j)^{(r)} - u_i^{(r)} u_j^{(r)} \tag{5}$$

We can likewise write the equation of motion for the filtered kinetic energy $E^{(r)} \equiv (1/2)(u_i^{(r)})^2$, obtaining

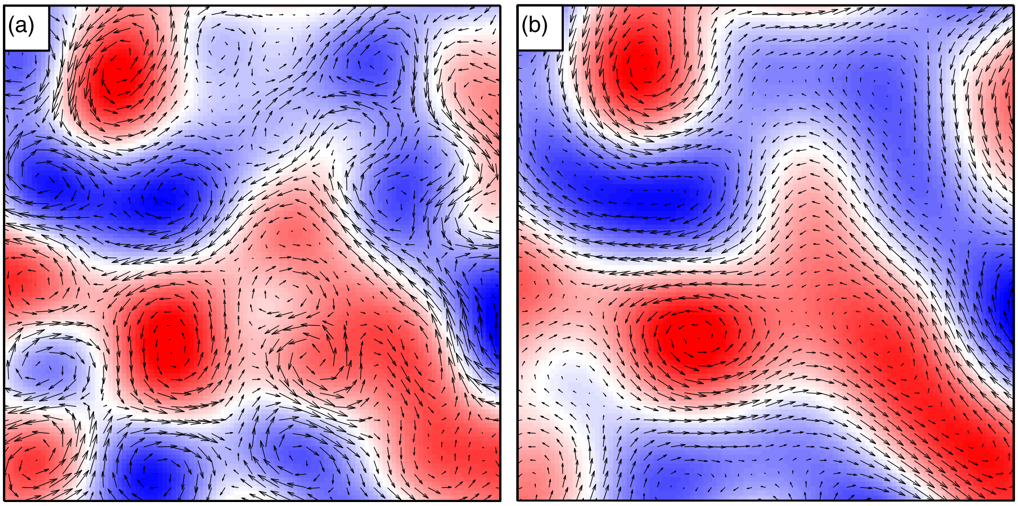

Fig. 4. (a) Full velocity (arrows) and vorticity (shading) field. (b) The same field filtered at $r = 2L_f$. Note that the small-scale structure is removed.

$$\frac{\partial E^{(r)}}{\partial t} = -\frac{\partial}{\partial x_i}\left[\frac{1}{\rho}u_i^{(r)}p^{(r)} + u_i^{(r)}E^{(r)} - \nu\frac{\partial E^{(r)}}{\partial x_i} + u_j^{(r)}\tau_{ij}^{(r)}\right] - 2\nu s_{ij}^{(r)}s_{ij}^{(r)} + \tau_{ij}^{(r)}s_{ij}^{(r)} \tag{6}$$

Here, $p^{(r)}$ is the filtered pressure, ρ is the fluid density, ν is the kinematic viscosity, $s_{ij}^{(r)} = (1/2)(\partial u_i^{(r)}/\partial x_j + \partial u_j^{(r)}/\partial x_i)$ is the rate of strain in the filtered field, and we are neglecting any forcing terms. The terms on the right-hand side in square brackets are associated with the spatial redistribution of $E^{(r)}$, since they are all total divergences. Aside from the term containing $\tau_{ij}^{(r)}$, they are all directly analogous to terms in the equation of motion for the full kinetic energy. The second term (proportional to the viscosity) represents the direct viscous dissipation of the filtered energy, since it is non-positive and is not a total divergence; again, the analogous term appears in the equation for the full energy. The final term, however, is qualitatively different. It cannot be written as a total divergence, and is therefore a source or sink term; its sign is not *a priori* determined. It does not appear in the equation for the full kinetic energy, and is therefore purely a result of the filtering operation. We can thus associate it with an energetic coupling to the removed scales. We define

$$\Pi^{(r)} \equiv -\tau_{ij}^{(r)}s_{ij}^{(r)} \tag{7}$$

as the spectral flux of energy through the scale r, coupling together all scales larger than r and all scales smaller than r. With this sign convention, $\Pi^{(r)} > 0$ implies transfer to smaller scales (since that decreases $\partial E^{(r)}/\partial t$) while $\Pi^{(r)} < 0$ implies transfer to larger scales (increasing $\partial E^{(r)}/\partial t$). Since r can be varied, we can thus study the way energy passes through each scale of interest in the flow field. This method has been used, for example, to show convincingly the existence of inverse energy cascades (over limited scale ranges) even at relatively low Reynolds numbers in quasi-2D turbulence experiments [68,72].

The true power of these filter-space techniques, however, is not that they allow the measurement of the flux of energy between scales. Rather, it is that the spectral flux $\Pi^{(r)}$ is spatially localized. Thus, we can measure spatial fields of the spectral energy flux, as shown in Fig. 5, and begin to try to correlate spectral activity with coherent structures defined using the velocity field. Or, we can try to use the spectral flux fields themselves to help us define coherent structures. One low-order way to do this is shown in Fig. 6, where we plot isocontours of the spectral flux in a space spanned by the two spatial dimensions of the flow and by filter scale. Although such an approach likely suffers from the same problems as does looking at contours of "large" vorticity, it suggests the possibility of expanding our definition of "coherent structures" to regions that are smooth and localized not only just in space and time but also in scale. In particular, such regions may be considered to be "eddies" in the sense of the Richardson cascade. Defining such structures precisely is a major and important challenge for turbulence research.

6. Summary and outlook

The concept of coherent structures has spurred activity in turbulence research for decades. Even simple structures like Eulerian vortices have proved conceptually useful in gaining qualitative insight into fluid flow. But to make progress towards a truly predictive decomposition into structures and a featureless background – the holy grail of the coherent-structure approach – we must be able to connect the structures we define to the essential dynamics of the flow. This task is much more difficult than simply defining plausible types of structures, and has thus received much less attention.

Here, I have briefly reviewed several kinds of coherent structures, both Eulerian and Lagrangian, as well as a promising framework for evaluating their dynamical role by studying their relationship to the local scale-to-scale energy transfer.

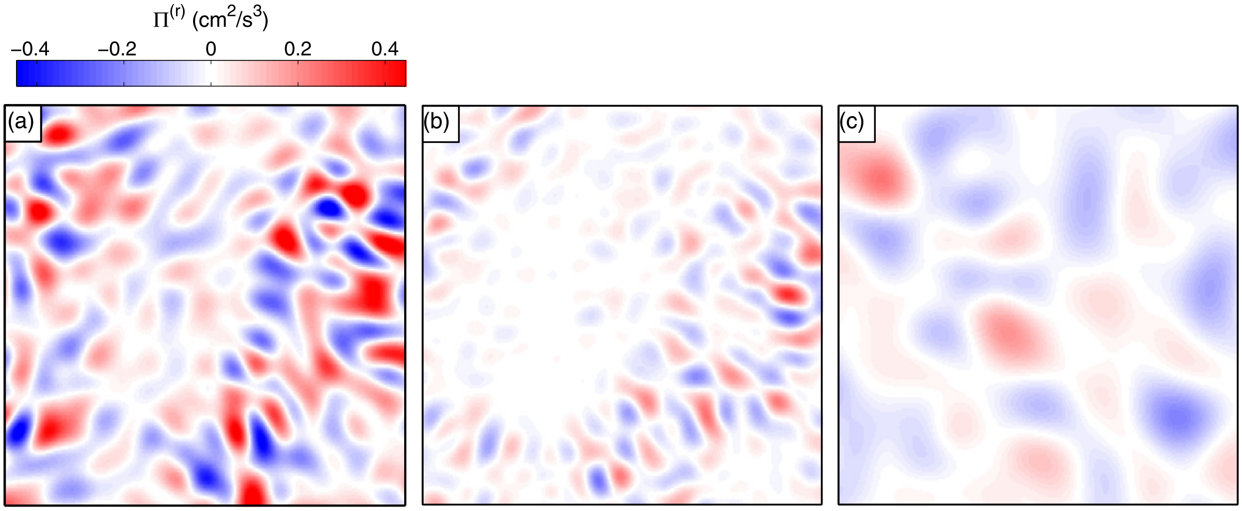

Fig. 5. Spectral flux fields through different scales for the same data as in Fig. 1. The flux is computed through (a) $r = L_f$, (b) $r = 0.65L_f$, and (c) $r = 2L_f$. The colormap is the same for each panel; flux is the most intense at L_f. Red regions show energy transfer to smaller scales, while blue regions show transfer to larger scales.

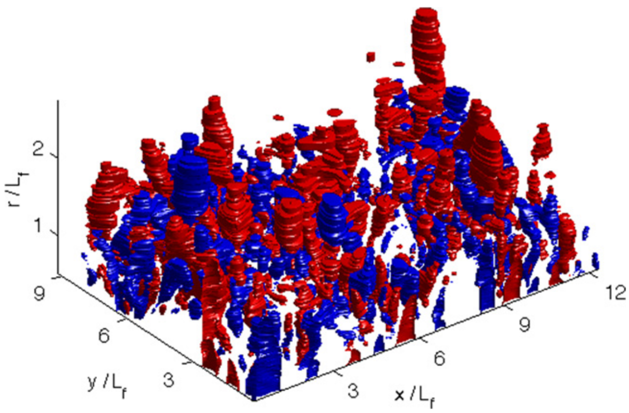

Fig. 6. Isocontours (at twice the root-mean-square value) of the spectral energy flux in a space spanned by the two spatial dimensions x and y and the filter scale r. Note how the flux varies smoothly in r (the stair-stepping is an artifact of the discreteness of our filter) and leads to objects that are compact in space and scale. These objects can perhaps be considered "eddies" in the sense of Richardson.

Although Eulerian structures are in many ways more appealing, since they are more directly connected to our raw observations, it seems clear that Lagrangian structures are more likely to be connected to the flow dynamics since they incorporate space and time together in a natural way. With new tools like filter-space techniques now available, fully evaluating the dynamical relevance of Lagrangian structures should be a fruitful area of exploration in the coming years. Additionally, such new tools may allow the definition of spatiotemporally coherent Eulerian structures that also play a defined role in the dynamics.

Acknowledgements

Thanks to D. Kelley, S. Merrifield, A. de Chaumont Quitry, and Y. Liao, who conducted the experiments and worked on the analyses discussed here. This work was supported by the US National Science Foundation under Grant No. DMR-0906245.

References

[1] S.B. Pope, Turbulent Flows, Cambridge University Press, Cambridge, England, 2000.
[2] M.C. Cross, P.C. Hohenberg, Pattern formation outside of equilibrium, Rev. Mod. Phys. 65 (1993) 851–1112.
[3] G.L. Brown, A. Roshko, On density effects and large structure in turbulent mixing layers, J. Fluid Mech. 64 (1974) 775–816.
[4] A.K.M.F. Hussain, Coherent structures and turbulence, J. Fluid Mech. 173 (1986) 303–356.
[5] P. Tabeling, Two-dimensional turbulence: a physicist approach, Phys. Rep. 362 (2002) 1–62.
[6] Y. Liao, D.H. Kelley, N.T. Ouellette, Effects of forcing geometry on two-dimensional weak turbulence, Phys. Rev. E 86 (2012) 036306.
[7] G. Boffetta, R.E. Ecke, Two-dimensional turbulence, Annu. Rev. Fluid Mech. 44 (2012) 427–451.

[8] D.H. Kelley, N.T. Ouellette, Onset of three-dimensionality in electromagnetically forced thin-layer flows, Phys. Fluids 23 (2011) 045103.
[9] T.H. Solomon, I. Mezić, Uniform resonant chaotic mixing in fluid flows, Nature 425 (2003) 376–380.
[10] H.J.H. Clercx, G.J.F. van Heijst, Two-dimensional Navier–Stokes turbulence in bounded domains, Appl. Mech. Rev. 62 (2009) 020802.
[11] T.P. Sapsis, N.T. Ouellette, J.P. Gollub, G. Haller, Neutrally buoyant particle dynamics in fluid flows: Comparison of experiments with Lagrangian stochastic models, Phys. Fluids 23 (2011) 093304.
[12] N.T. Ouellette, P.J.J. O'Malley, J.P. Gollub, Transport of finite-sized particles in chaotic flow, Phys. Rev. Lett. 101 (2008) 174504.
[13] D. Vella, L. Mahadevan, The "Cheerios" effect, Am. J. Phys. 73 (2005) 817–825.
[14] N.T. Ouellette, H. Xu, E. Bodenschatz, A quantitative study of three-dimensional Lagrangian particle tracking algorithms, Exp. Fluids 40 (2006) 301–313.
[15] N. Mordant, A.M. Crawford, E. Bodenschatz, Experimental Lagrangian probability density function measurement, Physica D 193 (2004) 245–251.
[16] A.K.M.F. Hussain, Coherent structures – reality and myth, Phys. Fluids 26 (1983) 2816–2850.
[17] J.C.R. Hunt, A. Wray, P. Moin, Eddies, stream, and convergence zones in turbulent flows, Technical Report CTR-S88, Center for Turbulence Research, 1988.
[18] G. Haller, An objective definition of a vortex, J. Fluid Mech. 525 (2005) 1–26.
[19] J. Jeong, F. Hussain, On the identification of a vortex, J. Fluid Mech. 285 (1995) 69–94.
[20] M. Rivera, X.-L. Wu, C. Yeung, Universal distribution of centers and saddles in two-dimensional turbulence, Phys. Rev. Lett. 87 (2001) 044501.
[21] L. Biferale, G. Boffetta, A. Celani, A. Lanotte, F. Toschi, Particle trapping in three-dimensional fully developed turbulence, Phys. Fluids 17 (2005) 021701.
[22] J. Bec, L. Biferale, M. Cencini, A.S. Lanotte, F. Toschi, Effects of vortex filaments on the velocity of tracers and heavy particles in turbulence, Phys. Fluids 18 (2006) 081702.
[23] A. Tsinober, Is concentrated vorticity that important? Eur. J. Mech. B/Fluids 17 (1998) 421–449.
[24] A.E. Perry, M.S. Chong, A description of eddying motion and flow patterns using critical-point concepts, Annu. Rev. Fluid Mech. 19 (1987) 125–155.
[25] S.C. Jana, J.M. Ottino, Chaos-enhanced transport in cellular flows, Phil. Trans. R. Soc. A 338 (1992) 519–532.
[26] D.A. Egolf, Dynamical dimension of defects in spatiotemporal chaos, Phys. Rev. Lett. 81 (1998) 4120–4123.
[27] M.S. Chong, A.E. Perry, B.J. Cantwell, A general classification of three-dimensional flow fields, Phys. Fluids A 2 (1990) 765–777.
[28] N.T. Ouellette, J.P. Gollub, Curvature fields, topology, and the dynamics of spatiotemporal chaos, Phys. Rev. Lett. 99 (2007) 194502.
[29] N.T. Ouellette, J.P. Gollub, Dynamic topology in spatiotemporal chaos, Phys. Fluids 20 (2008) 064104.
[30] W. Braun, F. De Lillo, B. Eckhardt, Geometry of particle paths in turbulent flows, J. Turbul. 7 (2006) 1.
[31] H. Xu, N.T. Ouellette, E. Bodenschatz, Curvature of Lagrangian trajectories in turbulence, Phys. Rev. Lett. 98 (2007) 050201.
[32] J.F. Foss, Surface selections and topological constraint evaluations for flow field analyses, Exp. Fluids 37 (2004) 883–898.
[33] T. Faber, J.C. Vassilicos, Acceleration-based classification and evolution of fluid flow structures in two-dimensional turbulence, Phys. Rev. E 82 (2010) 026312.
[34] S. Goto, J.C. Vassilicos, Self-similar clustering of inertial particles and zero-acceleration points in fully developed two-dimensional turbulence, Phys. Fluids 18 (2006) 115103.
[35] M. Chertkov, A. Pumir, B.I. Shraiman, Lagrangian tetrad dynamics and the phenomenology of turbulence, Phys. Fluids 11 (1999) 2394–2410.
[36] A. Pumir, B.I. Shraiman, M. Chertkov, Geometry of Lagrangian dispersion in turbulence, Phys. Rev. Lett. 85 (2000) 5324–5327.
[37] B.I. Shraiman, E.D. Siggia, Scalar turbulence, Nature 405 (2000) 639–646.
[38] A. Pumir, B.I. Shraiman, M. Chertkov, The Lagrangian view of energy transfer in turbulent flow, Europhys. Lett. 56 (2001) 379–385.
[39] H. Xu, N.T. Ouellette, E. Bodenschatz, Evolution of geometric structures in intense turbulence, New J. Phys. 10 (2008) 013012.
[40] L. Biferale, G. Boffetta, A. Celani, B.J. Devenish, A. Lanotte, F. Toschi, Multiparticle dispersion in fully developed turbulence, Phys. Fluids 17 (2005) 111701.
[41] H. Xu, A. Pumir, E. Bodenschatz, The pirouette effect in turbulent flows, Nat. Phys. 7 (2011) 709–712.
[42] S.T. Merrifield, D.H. Kelley, N.T. Ouellette, Scale-dependent statistical geometry in two-dimensional flow, Phys. Rev. Lett. 104 (2010) 254501.
[43] P. Castiglione, A. Pumir, Evolution of triangles in a two-dimensional turbulent flow, Phys. Rev. E 64 (2001) 056303.
[44] M.A.I. Khan, A. Pumir, J.C. Vassilicos, Kinematic simulation of turbulent dispersion of triangles, Phys. Rev. E 68 (2003) 026313.
[45] A. de Chaumont Quitry, D.H. Kelley, N.T. Ouellette, Mechanisms driving shape distortion in two-dimensional flow, EPL 94 (2011) 64006.
[46] S. Goto, J.C. Vassilicos, Particle pair diffusion and persistent streamline topology in two-dimensional turbulence, New J. Phys. 6 (2004) 65.
[47] G. Haller, Finding finite-time invariant manifolds in two-dimensional velocity fields, Chaos 10 (2000) 99–108.
[48] G. Haller, G. Yuan, Lagrangian coherent structures and mixing in two-dimensional turbulence, Physica D 147 (2000) 352–370.
[49] G. Haller, Distinguished material surfaces and coherent structures in three-dimensional fluid flows, Physica D 149 (2001) 248–277.
[50] M. Mathur, G. Haller, T. Peacock, J.E. Ruppert-Felsot, H.L. Swinney, Uncovering the Lagrangian skeleton of turbulence, Phys. Rev. Lett. 98 (2007) 144502.
[51] S.C. Shadden, F. Lekien, J.E. Marsden, Definition and properties of Lagrangian coherent structures from finite-time Lyapunov exponents in two-dimensional aperiodic flows, Physica D 212 (2005) 271–304.
[52] G.A. Voth, G. Haller, J.P. Gollub, Experimental measurements of stretching fields in fluid mixing, Phys. Rev. Lett. 88 (2002) 254501.
[53] S.C. Shadden, J.O. Dabiri, J.E. Marsden, Lagrangian analysis of fluid transport in empirical vortex ring flows, Phys. Fluids 18 (2006) 047105.
[54] F.J. Beron-Vera, M.J. Olascoaga, G.J. Goni, Oceanic mesoscale eddies as revealed by Lagrangian coherent structures, Geophys. Res. Lett. 35 (2008) L12603.
[55] E.T. Kai, V. Rossi, J. Sudre, H. Weimerskirch, C. Lopez, E. Hernandez-Garcia, F. Marsac, V. Garçon, Top marine predators track Lagrangian coherent structures, Proc. Natl. Acad. Sci. USA 106 (2008) 8245–8250.
[56] T. Peacock, J. Dabiri, Introduction to focus issue: Lagrangian coherent structures, Chaos 20 (2010) 017501.
[57] M.J. Olascoaga, G. Haller, Forecasting sudden changes in environmental pollution patterns, Proc. Natl. Acad. Sci. USA 109 (2012) 4738–4743.
[58] G. Haller, A variational theory of hyperbolic Lagrangian coherent structures, Physica D 240 (2011) 574–598.
[59] M. Farazmand, G. Haller, Computing Lagrangian coherent structures from their variational theory, Chaos 22 (2012) 013128.
[60] S. Parsa, J.S. Guasto, M. Kishore, N.T. Ouellette, J.P. Gollub, G.A. Voth, Rotation and alignment of rods in two-dimensional chaotic flow, Phys. Fluids 23 (2011) 043302.
[61] D.H. Kelley, N.T. Ouellette, Separating stretching from folding in fluid mixing, Nat. Phys. 7 (2011) 477–480.
[62] M.L. Falk, J.S. Langer, Dynamics of viscoplastic deformation in amorphous solids, Phys. Rev. E 57 (1998) 7192–7205.
[63] M.A. Green, C.W. Rowley, G. Haller, Detection of Lagrangian coherent structures in three-dimensional turbulence, J. Fluid Mech. 572 (2007) 111–120.
[64] L.F. Richardson, Weather Prediction by Numerical Process, Cambridge University Press, Cambridge, England, 1922.
[65] M. Germano, Turbulence: the filtering approach, J. Fluid Mech. 238 (1992) 325–336.
[66] S. Liu, C. Meneveau, J. Katz, On the properties of similarity subgrid-scale models as deduced from measurements in a turbulent jet, J. Fluid Mech. 275 (1994) 83–119.
[67] G.L. Eyink, Local energy flux and the refined similarity hypothesis, J. Stat. Phys. 78 (1995) 335–351.
[68] M.K. Rivera, W.B. Daniel, S.Y. Chen, R.E. Ecke, Energy and enstrophy transfer in decaying two-dimensional turbulence, Phys. Rev. Lett. 90 (2003) 104502.
[69] S. Chen, R.E. Ecke, G.L. Eyink, X. Wang, Z. Xiao, Physical mechanism of the two-dimensional enstrophy cascade, Phys. Rev. Lett. 91 (2003) 214501.
[70] S. Chen, R.E. Ecke, G.L. Eyink, M. Rivera, M. Wan, Z. Xiao, Physical mechanism of the two-dimensional inverse energy cascade, Phys. Rev. Lett. 96 (2006) 084502.
[71] Z. Xiao, M. Wan, S. Chen, G.L. Eyink, Physical mechanism of the inverse energy cascade of two-dimensional turbulence: a numerical investigation, J. Fluid Mech. 619 (2009) 1–44.
[72] D.H. Kelley, N.T. Ouellette, Spatiotemporal persistence of spectral fluxes in two-dimensional weak turbulence, Phys. Fluids 23 (2011) 115101.